Higher Mathematics

Ya. S. BUGROV / S. M. NIKOLSKY

differential equations multiple integrals series theory of functions of a complex variable

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ABOUT THE BOOK

The present textbook, together with three other books: **Fundamentals of Linear Algebra and Analytical Geometry, Differential and Integral Calculus,** and **A Collection of Problems,** form a four-book series entitled "Higher Mathematics".

The book consists of seven chapters. Chapter 1 is dedicated to ordinary differential equations and Chapter 2 deals with multiple integrals. Chapter 3 presents a study of vector analysis. Fourier series and Fourier integrals are discussed in Chapter 4. Chapter 5 treats equations of mathematical physics. Chapter 6 contains the theory of functions of a complex variable, and Chapter 7 operational calculus.

Each chapter opens with principal concepts and notions. As a rule, formal proofs of the theorems under consideration are given at the end of the chapter or section. This enables the reader to confine himself, in the case of necessity, to the beginnings of the chapters or sections.

The exercises should be regarded as an integral part of the book. There is a great deal more to be learned from attempting the exercises than can be obtained from a passive reading of the text.

ИЗДАТЕЛЬСТВО
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ВЫСШАЯ МАТЕМАТИКА

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С. М. Никольский

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by

Leonid Levant

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PREFACE TO THE ENGLISH EDITION

The present textbook, together with three other books: *Fundamentals of Linear Algebra and Analytical Geometry*, *Differential and Integral Calculus*, and *A Collection of Problems*, form a four-book series entitled "Higher Mathematics". This series covers the curriculum on higher mathematics for engineering students.

However, not all aspects of approximate methods are treated in our series; nor are the theory of probability and statistics dealt with.

The book consists of seven chapters. Chapter 1 is dedicated to ordinary differential equations and Chapter 2 deals with multiple integrals. Chapter 3 presents a study of vector analysis. Fourier series and Fourier integrals are discussed in Chapter 4. Chapter 5 treats equations of mathematical physics. Chapter 6 contains the theory of functions of a complex variable, and Chapter 7 operational calculus.

Each chapter opens with principal concepts and notions. As a rule, formal proofs of the theorems under consideration are given at the end of the chapter or section. This enables the reader to confine himself, in the case of necessity, to the beginnings of the chapters or sections.

In Chapters 4 and 5, when deriving formulas in a number of cases we confine ourselves only to physical considerations.

We should like to note that Chapter 6 and Chapter 7 may be read prior to Chapter 4. The latter requires no command of the theory of functions of a complex variable save for a rudimentary knowledge of complex numbers.

The exercises should be regarded as an integral part of the book. There is a great deal more to be learned from attempting the exercises than can be obtained from a passive reading of the text.

For those of our readers who want to acquire a more profound knowledge of mathematics we recommend the following books:

Fundamentals of Mathematical Analysis by V. A. Ilyin and E. G. Poznyak, edited by A. N. Tikhonov (Mir Publishers, Moscow, 1982);

Mathematical Analysis by L. D. Kudryavtsev (in Russian);

A Course of Mathematical Analysis by S. M. Nikolsky (Mir Publishers, Moscow, 1981).

We should like to express our gratitude to the members of the Chairs of Mathematics of the Moscow Institute of Steel and Alloys, and the Moscow Power Institute headed by Professor V. A. Trenogin and Professor S. I. Pokhozhaev, respectively, for favourably reviewing our books comprising the series "Higher Mathematics" and their valuable advice.

We would also like to thank Professors E. A. Volkov and A. F. Leontyev, and Assistant Professor A. F. Lapko for useful remarks and a favourable attitude towards our books.

In conclusion, we would like to express our gratitude to the English Department of Mir Publishers for the most efficient handling of the publication of our books. We are particularly grateful to L. E. Levant who once again showed himself to be a skilled and diligent translator.

February 1982,
Moscow

Yakov S. Bugrov
Sergei M. Nikolsky

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CHAPTER 1

ORDINARY DIFFERENTIAL EQUATIONS

Sec. 1.1. Problem Leading to a Differential Equation

Let a body having temperature θ_0 at time $t = 0$ be placed in a medium whose temperature is a ($\theta_0 > a$). It is required to find the law according to which the temperature of the given body changes with time. The sought-for temperature is a function of time which will be denoted by $\theta(t)$.

As is known from physics, the cooling rate of a body is proportional to the difference between the temperature of the body and ambient temperature. Taking into consideration that the function $\theta(t)$ is decreasing, by virtue of the mechanical sense of the derivative we obtain

$$\frac{d\theta(t)}{dt} = -k[\theta(t) - a], \tag{1}$$

where k is a proportionality factor.

Relationship (1) is a mathematical model of the physical process under consideration. It is called a *differential equation*, since, alongside with an unknown function $\theta(t)$, this equation contains its derivative. Differential equation (1) can describe other physical processes as well. For instance, radioactive decay is also described by equation (1) for $a = 0$.

The solution of equation (1) is readily guessed: $\theta(t) = Ce^{-kt} + a$, where C is an arbitrary constant. The value of this constant can be found from the condition $\theta(0) = \theta_0$, from which it follows that $\theta_0 = C + a$.

Thus, the required solution has the form

$$\theta(t) = (\theta_0 - a)\, e^{-kt} + a.$$

Sec. 1.2. General

When studying physical phenomena one frequently fails to find immediately a law relating independent variables and the sought-for function, but can be successful in estab-

lishing the relationship between this function and its derivatives expressed by a differential equation.

If the required function depends on one variable, then such a differential equation is called *ordinary*. An arbitrary ordinary differential equation of order n has the following form:

$$F(x, y, y', \ldots, y^{(n)}) = 0. \tag{1}$$

Here F is a given (known) function of $n + 2$ variables usually satisfying certain conditions of continuity and differentiability on which we are not going to dwell now, and $y = y(x)$ (a function of x) is the solution of the differential equation to be found.

A *solution of a differential equation* of order n is defined as a function $y(x)$ having on some interval (a, b) derivatives $y'(x)$, $y''(x)$, $\ldots$, $y^{(n)}(x)$ up to order n included and satisfying this equation. This means that the identity with respect to x is fulfilled:

$$F(x, y(x), y'(x), \ldots, y^{(n)}(x)) \equiv 0, \quad x \in (a, b).$$

Generally speaking, to each solution there corresponds its own interval. Of course, if a function $y(x)$ defined on the interval (a, b) is a solution of differential equation (1), then this function considered on the interval (c, d) belonging to (a, b) is also a solution of equation (1).

In the sections to follow we shall deal with differential equations defined by real functions F and find their real solutions $y(x)$. For the sake of brevity, we shall omit the word 'real'. Later on, when studying linear differential equations, we shall also need their complex solutions. But this will be considered below.

Thus, real solutions $y(x)$, $x \in (a, b)$, of an ordinary differential equation will be called simply solutions of this equation.

By definition, a solution of an ordinary differential equation of order n is a function $y(x)$, continuous on some interval (a, b) together with its derivatives up to order $n - 1$ included, which has, in addition, a derivative $y^{(n)}(x)$ of order n on (a, b). We shall assume that this last derivative is also continuous on (a, b) without stipulating this each time.

The graph of a solution of a differential equation of order n will be called the *integral curve* of this equation (see the Note in the next section).

However, we shall call a solution of a differential equation an integral curve and, vice versa, an integral curve will be called a solution.

Since this chapter is dedicated only to ordinary differential equations, the word 'ordinary' will be sometimes omitted.

The equations

$$y''' + 2y' + y = \sin x, \quad (y'')^2 + 1 = 0,$$
$$y'' + (y')^2 = 0, \quad y' + ky = \cos x$$

are examples of ordinary differential equations, the first of them being of the third order, the second and the third of the second order, and the forth of the first order.

Note in passing that it is directly seen that the second equation has no real solutions at all.

There exists the term 'to integrate a differential equation'. This means that it is required to find certain solutions of a given differential equation. The finding of a solution of a differential equation is always connected with the necessity to integrate the functions entering into this equation.

We shall begin our study with a first-order differential equation

$$F(x, y, y') = 0. \tag{2}$$

As a rule, we shall suppose that the function $F(x, y, z)$ is defined in some domain of the three-dimensional space Ω and is continuous on Ω together with its partial derivatives $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$. In particular, Ω can represent the entire three-dimensional space of points (x, y, z).

Let us recall that by a *solution* or *particular solution* of differential equation (2) we mean any real continuously differentiable function $y = y(x)$ defined on some interval (a, b) which satisfies this equation:

$$F(x, y(x), y'(x)) \equiv 0, \quad x \in (a, b),$$
$$(x, y(x), y'(x)) \in \Omega.$$

Generally speaking, each solution has its own interval on which it is defined.

Two algebraic equations

$$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0 \tag{3}$$

are called equivalent on a domain Ω of points (x, y, z) if from the fact that a point $(x, y, z) \in \Omega$ satisfies one of these equations it follows that it also satisfies the other.

Accordingly, two differential equations

$$F_1(x, y, y') = 0, \quad F_2(x, y, y') = 0$$

are called *equivalent* on a domain Ω if algebraic equations (3) are equivalent on Ω.

Thus, in this case a solution $y(x)$, $x \in (a, b)$, $(x, y(x), y'(x)) \in \Omega$, of one of the differential equations is automatically a solution of the other.

However, differential equations equivalent on a domain Ω are regarded as one and the same equation.

When transforming a differential equation, one should watch that the new differential equation obtained as a result of the transformation be equivalent (on Ω!) to the original one. Or, in any case, one should note which of the solutions may disappear or crop up after a certain transformation.

Let us mention here a problem called *Cauchy's* problem* for a first-order differential equation. It reads: It is required to find the solution $y = y(x)$ of a given differential equation, satisfying the initial condition

$$y(x_0) = y_0,$$

where (x_0, y_0) is a given point of the xy-plane.

Of course, in each individual case Cauchy's problem can either have a solution or have no solution.

If Cauchy's problem has a solution it is important to find out whether it is unique. We are going to note here an important fact which will be proved in Sec. 1.6: for a first-order differential equation in the form solved with respect to y'

$$\frac{dy}{dx} = f(x, y) \qquad ((x, y) \in G)$$

---

* Cauchy, Augustin Louis (1789-1857). Outstanding French mathematician.

Cauchy's problem has a unique solution for any point (x_0, y_0) of a domain G of the xy-plane if the function $f(x, y)$ defined on this domain is continuous together with its partial derivative $\frac{\partial f}{\partial y}$.

Of course, the uniqueness of the solution of Cauchy's problem should be understood in the sense that if $y(x)$ and $y_1(x)$ are its solutions satisfying one and the same initial condition ($y(x_0) = y_1(x_0) = y_0$) and defined on the intervals (a, b) and (c, d), respectively, then $y(x) = y_1(x)$ on the intersection of these intervals.

Example 1. A simplest first-order differential equation has the form

$$y' = f(x) \qquad (a < x < b), \tag{4}$$

where $f(x)$ is a continuous function on some interval (a, b).

It follows from the theory of indefinite integrals that any solution of this differential equation can be written in the following way:

$$y = \int f(x)\, dx + C,$$

where the first term on the right is an indefinite integral of $f(x)$, i.e. a primitive function of $f(x)$ on (a, b):

$$\psi(x) = \int f(x)\, dx,$$

the second term being an arbitrary constant C $(-\infty < C < \infty)$.

Thus, any solution of differential equation (4) is determined by the equation

$$y = \psi(x) + C \qquad (a < x < b), \tag{5}$$

where $\psi(x)$ is some primitive of $f(x)$ on (a, b) and C is an arbitrary constant known as the parameter of the family of solutions.

To every value of the parameter C there corresponds an individual (particular) solution of differential equation (4), and any solution of this equation can be obtained as a particular solution of family (5) for a certain value of C.

Differentiating equality (5) with respect to x, we obtain the original differential equation (4). Due to this property,

equality (5) containing an arbitrary constant C is called the general integral of differential equation (4).

Cauchy's problem for differential equation (4) is solved in a unique way for the initial condition $y(x_0) = y_0$, where (x_0, y_0) is any point from the strip $\{a < x < b, -\infty < y < \infty\}$ of the xy-plane. To solve it, we substitute the point (x_0, y_0) into general integral (5) and find the constant C:

$$y_0 = \psi(x_0) + C,$$

$$C = y_0 - \psi(x_0).$$

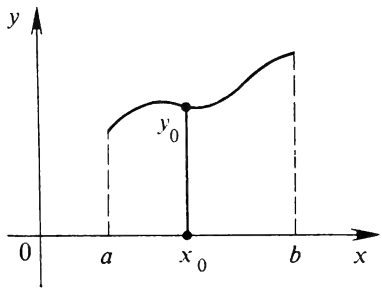

Fig. 1.

Hence we obtain

$$y - y_0 = \psi(x) - \psi(x_0).$$

This is just the solution (integral curve) of differential equation (4) passing through the point (x_0, y_0) (Fig. 1).

Example 2. Consider the differential equation

$$y' = ky \quad (-\infty < x, \ y < \infty), \tag{6}$$

where k is a given constant.

As is readily checked, the function

$$y = Ce^{kx} = C \exp(kx) \quad (-\infty < x < \infty) \tag{7}$$

is a solution of differential equation (6) for any value of the parameter C. We are not going to explain now how one can arrive logically at this family of solutions which depends on an arbitrary constant C (see the next section).

Let us differentiate equality (7) with respect to x:

$$y' = Ck \exp(kx) \quad (-\infty < x < \infty). \tag{8}$$

Now eliminate the parameter C from both equalities (7) and (8), i.e. find C from one of them and substitute into the other. This, obviously, will result in the original differential equation (6) obtained once again.

By virtue of this property, equality (7) is called the general integral of differential equation (6).

Let us give the definition of the general integral of a first-order differential equation.

Suppose we are given a first-order differential equation

$$F(x, y, y') = 0, \tag{2}$$

where the function $F(x, y, z)$ and its partial derivatives $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ are continuous on a domain Ω of points (x, y, z) of three-dimensional space.

The *general integral of differential equation* (2) is defined as the equality

$$\Phi(x, y, C) = 0, \tag{9}$$

where the function $\Phi(x, y, z)$ is continuously differentiable on some domain of points (x, y, z) and has the following property: differentiating equality (9) with respect to x (setting formally $y = y(x)$):

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} y' = 0, \tag{10}$$

and eliminating C from equations (9) and (10), we obtain a differential equation equivalent to equation (2).

Equation (2) is also called the differential equation of the family of functions (9) depending on the parameter C.

Example 3. Consider the family of functions

$$y = (x - C)^3 \quad (-\infty < x < \infty), \tag{11}$$

depending on an arbitrary parameter C.

Differentiate (11) with respect to x:

$$y' = 3(x - C)^2 \quad (-\infty < x < \infty) \tag{12}$$

and cube the obtained equality:

$$(y')^3 = 27(x - C)^6. \tag{13}$$

As is readily seen, we have obtained a differential equation which is equivalent to equation (12). But then from (11) and (13) there follows the differential equation

$$y'^3 - 27y^2 = 0. \tag{14}$$

It is easily verified that any function (11) satisfies this equation. However, this follows from the fact that differential equation (14) is the result of elimination of the parameter C from equalities (11) and (13).

We have proved that equality (11) containing an arbitrary parameter C is the general integral of differential equation (14).

In later sections we shall study certain types of first-order differential equations and indicate the methods for their solution which will lead to families of solutions depending on one parameter C. These families are usually the general integrals of respective differential equations.

There arises a question whether the general integral of a given first-order differential equation contains all the solutions of this equation for any values of the parameter C. Generally speaking, this is not so. But this automatically takes place if the general integral of a first-order differential equation can be written in the form solved with respect to C:

$$\Psi(x, y) = C, \tag{15}$$

the left-hand side of equation (15) being a *continuously differentiable* function. Namely, the following theorem holds true.

Theorem 1. *Let there be given a differential equation*

$$F(x, y, y') = 0, \quad (x, y, y') \in \Omega, \tag{2'}$$

where the function $F(x, y, z)$ *together with its partial derivatives* $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ *are continuous on a domain* Ω *of the space* (x, y, z), *and let equality* (15) *be its general integral, where* $\Psi(x, y)$ *is a function continuously differentiable on some domain of the xy-plane.*

Then, if

$$y = y(x), \quad (x, y(x), y'(x)) \in \Omega,$$

is a continuously differentiable on (a, b) *solution of equation* (15) *for some value of* C, *it is necessarily a solution of differential equation* (2') *and, conversely, any solution of differential equation* (2') *satisfies equation* (15) *on the interval on which it is defined for some constant* C.

Proof. Let $y = y(x)$, $x \in (a, b)$ be a continuously differentiable solution of equation (15) for the constant C_0:

$$\Psi(x, y(x)) = C_0, \quad x \in (a, b).$$

We differentiate this identity with respect to x:

$$\frac{\partial \Psi}{\partial x}(x, y(x)) + \frac{\partial \Psi}{\partial y}(x, y(x))\, y'(x) = 0, \quad x \in (a, b). \tag{16}$$

This shows that the function $y(x)$ is a solution of the differential equation

$$\frac{\partial \Psi}{\partial x}(x, y) + \frac{\partial \Psi}{\partial y}(x, y)\, y' = 0, \tag{17}$$

and, consequently, a solution of differential equation (2′) which is equivalent to equation (17) on Ω.

Conversely, let $y(x)$, $a < x < b$, be a solution of differential equation (2′) and, consequently, of equation (17), i.e. let there be fulfilled identity (16) which can be written as

$$\frac{d}{dx} \Psi(x, y(x)) \equiv 0, \quad x \in (a, b).$$

Integrating it from x_0 to x, where x_0, $x \in (a, b)$, we obtain

$$0 \equiv \int_{x_0}^{x} \frac{d}{dx} \Psi(x, y(x))\, dx = \Psi(x, y(x)) - \Psi(x_0, y(x_0))$$

$$= \Psi(x, y(x)) - C_0$$

$$(C_0 = \Psi(x_0, y(x_0))),$$

i.e. the function $y(x)$ satisfies the equation $\Psi(x, y) = C_0$ on (a, b).

Note to Example 2. The general integral of the differential equation

$$y' = ky \qquad (-\infty < y < \infty) \tag{6′}$$

solved with respect to C (see (7)) has the form

$$ye^{-kx} = C \qquad (-\infty < x, y < \infty). \tag{18}$$

Since the left-hand side of this equation has continuous partial derivatives on the entire xy-plane, by Theorem 1, the general integral (18) contains all the solutions of differential equation (6′) for different constants C.

To solve Cauchy's problem for differential equation (6′) under the initial condition $y(x_0) = y_0$, we substitute (x_0, y_0) into (18) and find $C = C_0$:

$$y_0 e^{-kx_0} = C_0.$$

The solution of Cauchy's problem has the form

$$ye^{-kx} = y_0 e^{-kx_0}$$

or

$$y = y_0 e^{k(x-x_0)} \quad (-\infty < x < \infty).$$

Note to Example 3. The differential equation in this example can be written in the form

$$F(x, y, y') = 0, \tag{19}$$

where the function

$$F(x, y, y') = y'^3 - 27y^2$$

is continuously differentiable in the entire space (x, y, z) which will be denoted by Ω.

We already know that the general integral of this equation has the form

$$y = (x - C)^3 \quad (-\infty < x < \infty). \tag{11}$$

Solving equation (11) with respect to C, we obtain

$$\Psi(x, y) = C, \quad \Psi(x, y) = x - y^{1/3}. \tag{11'}$$

The partial derivative of the function Ψ with respect to y is not existent on the axis $y = 0$, therefore the conditions of Theorem 1 are not fulfilled. But then it is impossible to guarantee that any solution of differential equation (19) enters into its general integral for some C.

Figure 2 represents a family of cubical parabolas (11) for different values of C. Each of these parabolas is an integral curve of differential equation (19). But there are other integral curves, for instance, the curve shown in Fig. 2 by a heavily drawn line:

$$y(x) = \begin{cases} (x-\alpha)^3, & x \leqslant \alpha, \\ 0, & \alpha < x < \beta, \\ (x-\beta)^3, & \beta \leqslant x. \end{cases}$$

Thus, differential equation (19) has a general integral (11) defined in the entire xy-plane, but it does not contain all the solutions of this equation for different values of C. There is an infinitude of solutions corresponding to the pairs of numbers (α, β), where $\alpha < \beta$, which are not obtained from family (11) for a certain value of C.

But if differential equation (19) is considered for positive y ($y > 0$), i.e. if it is assumed that the function $F(x, y, z)$ is defined in the half-space $y > 0$ which will be denoted by Ω_+, then the general integral

$$\Psi(x, y) = x - y^{1/3} = C,$$

$$-\infty < C < \infty, \quad -\infty < x < \infty, \quad 0 < y < \infty,$$

is defined by the function $\Psi(x, y)$ which is continuously differentiable in Ω_+. Therefore, in the present case, Theo-

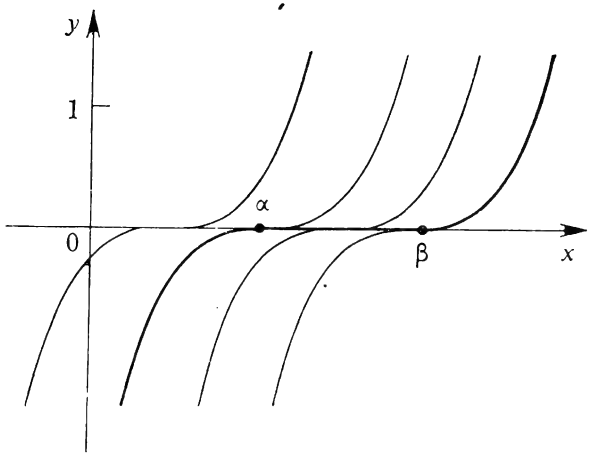

Fig. 2.

rem 1 is applicable, and the general integral contains for different C all the solutions of the differential equation belonging to the upper half-space

$$\{-\infty < x < \infty, \quad 0 < y < \infty\}.$$

A similar fact also takes place for the domain Ω_- of points (x, y, z) where $y < 0$.

Note that the differential equation in the form solved with respect to the derivative

$$\frac{dy}{dx} = f(x, y) \tag{20}$$

establishes an explicit relation between the coordinates of the point $M = (x, y)$ and the slope of the tangent $\frac{dy}{dx}$ to the

integral curve at this point (Fig. 3):

$$\tan \alpha = \frac{dy}{dx} = f(x, y).$$

If the function $f(x, y)$ is defined in some domain Ω of the plane, then to every point $M \in \Omega$ there corresponds a direc-

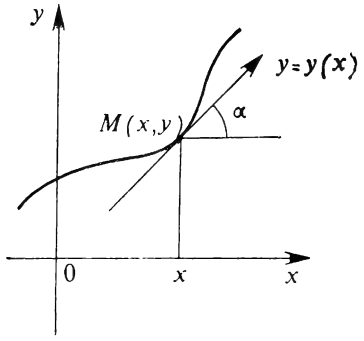

Fig. 3.

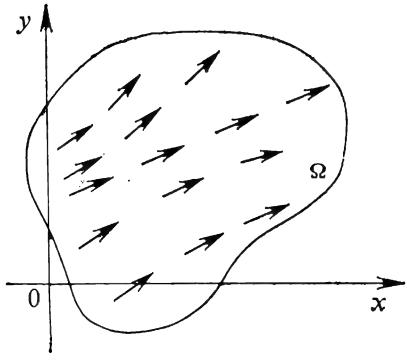

Fig. 4.

tion whose slope is equal to $f(x, y)$. Indicating this direction by the unit vector passing through the point M, we obtain a *field of tangent directions* in Ω or the direction field (Fig. 4).

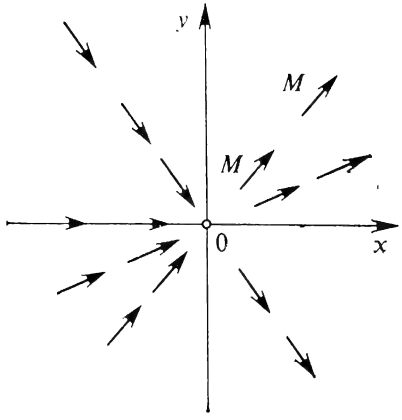

Fig. 5.

The integral curves of equation (20) are curves for which the mentioned directions are directions of tangent lines. To solve a differential equation means to find the curves the directions of tangents to which at every point coincide with the direction of the field. Of course, in this case the curves belong to the domain Ω.

Example 4. $y' = y/x$.

The right-hand member of this equation is defined on the set Ω of all points of the xy-plane except for the points of the y-axis. If the points $M = (x, y) \in \Omega$ lie on the straight

line $y = kx$, then for them

$$\tan \alpha = f(x, y) = f(x, kx) = \frac{kx}{x} = k$$

$$(x \neq 0),$$

i.e. the direction field has the form shown in Fig. 5.

In this case the direction of the straight line $y = kx$ coincides with the direction of the field at every point of

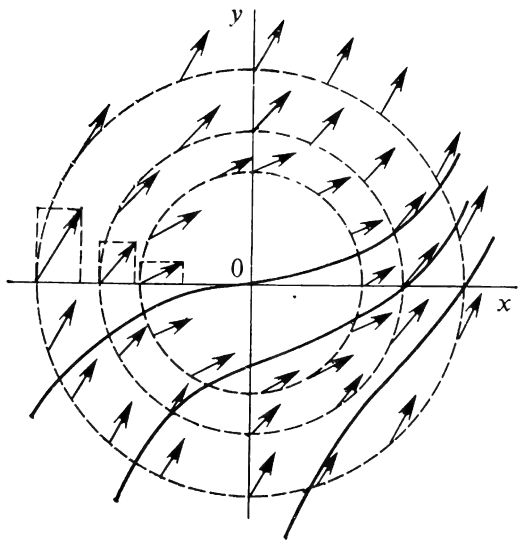

Fig. 6.

this line, consequently, the integral curves are rays (without the point (0, 0)) emanating from the origin which are not parallel to the y-axis.

For the construction of the direction field it is convenient to consider loci of points at which the lines tangent to integral curves retain constant direction. Such loci of points are called *isoclines* (line of equal inclination).

Example 5. $y' = \sqrt{x^2 + y^2}$; $\sqrt{x^2 + y^2} = k$ is the equation of the isocline corresponding to a definite value of k ($y' = k$), i.e. this is a circle of radius k (Fig. 6).

Knowing the isoclines of a differential equation, one can easily sketch its integral curves.

Sec. 1.3. Simplest First-order Differential Equations

Let $M(x, y)$ and $N(x, y)$ be two functions continuous in some domain Ω of the xy-plane.

The expression

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad ((x, y) \in \Omega) \qquad (1)$$

is called a *differential equation of the first order.*

Actually, expression (1) combines two first-order differential equations: with respect to the function $y(x)$ and with respect to the function $x(y)$.

In the first case by the solution of equation (1) we understand the function $y = y(x)$ defined on some interval (a, b) (dependent on it) which has a continuous derivative and satisfies equation (1)

$$M(x, y(x))\,dx + N(x, y(x))\,y'(x)\,dx = 0$$

$$(x \in (a, b), \quad (x, y(x)) \in \Omega).$$

Since the differential dx of the independent variable x is not equal to zero, we can reduce this equation by dx and obtain that $y(x)$ satisfies the first-order differential equation written in the ordinary form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \qquad ((x, y) \in \Omega). \qquad (2)$$

With respect to solutions of the form $y = y(x)$, differential equations (1) and (2) are equivalent.

Reasoning in an analogous way, we obtain that, with respect to solutions of the form $x = x(y)$, differential equation (1) is equivalent to the following one:

$$M(x, y)\frac{dx}{dy} + N(x, y) = 0 \qquad ((x, y) \in \Omega). \qquad (3)$$

Let us study in more detail differential equation (2) (with respect to y).

Let a function $N(x, y)$ be different from zero everywhere on Ω ($N(x, y) \neq 0$, $\forall (x, y) \in \Omega$) Then, by virtue of its continuity in a connected set Ω, it is either positive or negative everywhere on Ω. In this case equation (2) can be

written in the form solved with respect to $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}, \quad (x, y) \in \Omega, \tag{2'}$$

i.e. equations (2) and (2′) are equivalent on Ω. And if the function $N(x, y)$ is equal to zero at some points of Ω, then equations (2) and (2′) will be equivalent only on the portion ω of the domain Ω, where the function $N(x, y)$ is different from zero.

Let the function N vanish at the point $(x_0, y_0) \in \Omega$ ($N(x_0, y_0) = 0$). And if $M(x_0, y_0) \neq 0$, then equation (2), obviously, has no solution passing through this point, since the second term in the left-hand side of (2) is equal to zero for $x = x_0$, $y = y_0$, and the first one, by the hypothesis, is not equal to zero.

And if, alongside with the equality $N(x_0, y_0) = 0$, the equality $M(x_0, y_0) = 0$ is also fulfilled, then one or several or even an infinite number of solutions can pass through the point (x_0, y_0).

A similar remark can also be made with respect to equation (3). The only thing which is necessary is to interchange x and y, and also M and N in the corresponding argument.

Let us also consider the case when both functions M and N are different from zero everywhere on Ω. In this case the right-hand side of equation (2′) is also different from zero everywhere on Ω and has one and the same sign. But then the solution $y(x)$ of differential equation (2′) has the derivative $y'(x)$ of the same sign. This shows that the solution $y = y(x)$ is strictly monotone on the interval (a, b) on which it is defined. But then it has an inverse continuously differentiable function $x = x(y)$ on some interval (c, d). And

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{-\frac{M}{N}} = -\frac{N}{M},$$

which shows that the inverse function satisfies differential equation (3).

Thus, we have obtained that if both functions $M(x, y)$ and $N(x, y)$ are different from zero everywhere on Ω, then any solution of equation (1) of the form $y = y(x)$ has an

inverse function $x = x(y)$ which is also a solution of this equation, but of the form $x = x(y)$.

I. Equations with Separated Variables. Equation (1) is called a first-order differential equation *with separated variables* if

$$M(x, y) = \varphi(x), \quad x \in (a, b),$$
$$N(x, y) = \psi(y), \quad y \in (c, d).$$

It has the form

$$\varphi(x)\,dx + \psi(y)\,dy = 0. \tag{4}$$

We shall assume that $\varphi(x)$ and $\psi(y)$ are continuous functions. Let $y = y(x)$ be a solution of differential equation (4) in the rectangle

$$\Delta = \begin{Bmatrix} a < x < b \\ c < y < d \end{Bmatrix}$$

defined on some interval $(\alpha, \beta) \subset (a, b)$.

Then the following identity takes place:

$$\varphi(x)\,dx = -\psi[y(x)]\,dy(x), \quad x \in (\alpha, \beta),$$

whence, by integrating, we obtain

$$\int \varphi(x)\,dx = -\int \psi[y(x)]\,dy(x) + C_1 = -\int \psi(y)\,dy + C.$$

Here the integrals $\int \varphi(x)\,dx$ and $\int \psi(y)\,dy$ are some chosen primitives of $\varphi(x)$ and $\psi(y)$:

$$\int \varphi(x)\,dx = \Phi(x), \quad a < x < b,$$

$$\int \psi(y)\,dy = \Psi(y), \quad c < y < d;$$

in the second equality a change of variable $y(x) = y$ is performed in the indefinite integral (see our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 5.2, Mir Publishers, Moscow, 1982): the constant C depends on the solution $y(x)$.

Thus, any solution $y(x)$ of our differential equation in the indicated rectangle satisfies the equation

$$\int \varphi(x)\,dx + \int \psi(y)\,dy = C$$

for some constant C, or the equation

$$\Phi(x) + \Psi(y) = C. \tag{5}$$

The left-hand side of equality (5) is a function $F(x, y)$ which is continuously differentiable in the rectangle

$$\Delta = \{a < x < b, \quad c < y < d\},$$

with the properties

$$\frac{\partial F}{\partial x} = \varphi(x), \qquad \frac{\partial F}{\partial y} = \psi(y).$$

If we differentiate formally (5) with respect to x, assuming that $y = y(x)$, then we obtain

$$\varphi(x) + \psi(y)\frac{dy}{dx} = 0,$$

i.e. the original differential equation (4).

Thus, equation (5) is the general integral of differential equation (4) for its solutions of the form $y = y(x)$. According to Theorem 1 proved in the preceding section, all continuously differentiable on (α, β) solutions $y = y(x)$ of equation (5) for any constants C are solutions of differential equation (4) of the form $y = y(x)$ and conversely. However, the converse statement we have proved directly.

Reasoning in a similar way, interchanging x and y, we again obtain equality (5), but this time it will be the general integral containing all possible solutions of the form $x = x(y)$, $y \in (\gamma, \delta) \subset (c, d)$, of our differential equation (4).

Hence, equality (5) is the general integral of differential equation (4) both for solutions of the form $y = y(x)$ and for solutions of the form $x = x(y)$.

Example 1. $x^2\,dx = y\,dy$, $\Delta = \{a < x < b, \quad c < y < d\}$. $\int x^2\,dx - \int y\,dy = C$; $\frac{x^3}{3} - \frac{y^2}{2} = C$ — the general integral.

Example 2. $e^{x^2}\,dx = e^{y^2}\,dy$; $\int e^{x^2}\,dx - \int e^{y^2}\,dy = C$.

These integrals cannot be expressed in terms of elementary functions. Still, from the point of view of the theory of differential equations, we regard the problem as solved.

II. Equations with Variables Separable. If $M(x, y) = \varphi_1(x)\psi_1(y)$, $N(x, y) = \varphi_2(x)\psi_2(y)$, then equation (1)

is called an equation *with variables separable*:

$$\varphi_1(x)\,\psi_1(y)\,dx + \varphi_2(x)\,\psi_2(y)\,dy = 0. \tag{6}$$

For those (x, y) for which $\varphi_2(x)\cdot\psi_1(y) \neq 0$, we divide both sides of (6) by this product. Then we shall obtain the equation with separated variables

$$\frac{\varphi_1(x)}{\varphi_2(x)}\,dx + \frac{\psi_2(y)}{\psi_1(y)}\,dy = 0.$$

The general integral of this equation is found as in I. But there can also exist solutions passing through points (x_0, y_0) and satisfying the equation $\varphi_2(x_0)\,\psi_1(y_0) = 0$.

III. Homogeneous Equations. A function $M(x, y)$ is called *homogeneous of degree m* if for any x, y and $t > 0$ the following equality is fulfilled:

$$M(tx, ty) = t^m M(x, y).$$

If the functions $M(x, y)$ and $N(x, y)$ are homogeneous of one and the same degree m, then the differential equation

$$M\,dx + N\,dy = 0 \tag{7}$$

is said to be *homogeneous*.

It can be transformed in the following way:

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} = -\frac{M\left(x, |x|\frac{y}{|x|}\right)}{N\left(x, |x|\frac{y}{|x|}\right)}$$

$$= -\frac{|x|^m M\left(\pm 1, \pm\frac{y}{x}\right)}{|x|^m N\left(\pm 1, \pm\frac{y}{x}\right)} = f\left(\frac{y}{x}\right),$$

i.e.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), \tag{8}$$

where f is a function of one variable.

Let us introduce, instead of y, a new function z (of x) with the aid of the substitution

$$y = x\cdot z, \qquad \frac{dy}{dx} = x\,\frac{dz}{dx} + z.$$

Then

$$x\frac{dz}{dx}+z=f(z)$$

or

$$\frac{dz}{f(z)-z}=\frac{dx}{x}.$$

Consequently,

$$\ln\left|\frac{x}{C}\right|=\int\frac{dz}{f(z)-z}\quad(C\neq 0)$$

or

$$x=C\exp\left(\int\frac{dz}{f(z)-z}\right),$$

where $C\neq 0$ is an arbitrary constant.

Let us consider a more general equation, as compared with (8):

$$\frac{dy}{dx}=x^{\alpha-1}f\left(\frac{y}{x^{\alpha}}\right). \tag{9}$$

It can be solved by means of the substitution

$$y=x^{\alpha}z,\qquad \frac{dy}{dx}=\alpha x^{\alpha-1}z+x^{\alpha}\frac{dz}{dx};$$

then

$$\alpha x^{\alpha-1}z+x^{\alpha}\frac{dz}{dx}=x^{\alpha-1}f(z),$$

$$x^{\alpha}\frac{dz}{dx}=x^{\alpha-1}[f(z)-\alpha z],$$

$$\frac{dz}{f(z)-\alpha z}=\frac{dx}{x},$$

$$\ln\left|\frac{x}{C}\right|=\int\frac{dz}{f(z)-\alpha z}\quad(C\neq 0),$$

$$x=C\exp\int\frac{dz}{f(z)-\alpha z}, \tag{10}$$

where $C\neq 0$ is an arbitrary constant.

Example 3. $(x^2+y^2)\,dx+xy\,dy=0$.

The given equation is homogeneous, since the functions

$$M(x,\ y)=x^2+y^2,\quad N(x,\ y)=xy$$

are homogeneous of degree $m=2$. Let us make the substitution: $y=zx$, $dy=z\,dx+x\,dz$. Then the equation will

be rewritten as

$$(x^2 + z^2x^2)\,dx + x^2z\,(z\,dx + x\,dz) = 0$$

or

$$(1 + 2z^2)\,dx + zx\,dz = 0.$$

Separating the variables, we obtain

$$\frac{dx}{x} = -\frac{z\,dz}{1+2z^2}, \quad \ln\left|\frac{x}{C}\right| = -\frac{1}{4}\ln(1+2z^2),$$

$$x = \frac{C}{\sqrt[4]{1+2z^2}}.$$

Since $z = y/x$, we have

$$x^4 = \frac{C^4x^2}{x^2+2y^2}, \quad 2y^2 + x^2 = \frac{C^4}{x^2}, \quad y = \pm\sqrt{\frac{C^4}{2x^2} - \frac{x^2}{2}}.$$

Example 4.

$$y' = Ax^\gamma + By^\nu, \tag{11}$$

$$y' = x^\gamma\left[A + B\frac{y^\nu}{x^\gamma}\right] = x^\gamma\left[A + B\left(\frac{y}{x^{\gamma/\nu}}\right)^\nu\right].$$

This equation is a particular case of (9) if

$$\gamma + 1 = \frac{\gamma}{\nu}. \tag{12}$$

For $\gamma = -2$ and $\nu = 2$ (condition (12) is fulfilled) equation (11) has the form

$$\frac{dy}{dx} = Ax^{-2} + By^2,$$

and its solution is written by formula (10), where

$$f(z) = A + Bz^2, \quad \alpha = \gamma + 1 = -1.$$

The obtained equation is a particular case of the *Riccati* equation*

$$y' = By^2 + R(x),$$

which, as is known for $R(x) \not\equiv Ax^{-2}$ is not solvable in quadratures.

Example 5. $xy\,dy - (x^4 + y^2)\,dx = 0.$

* Riccati, Count Jacopo Francesco (1676-1754). Italian mathematician.

We have:

$$\frac{dy}{dx}=\frac{x^4+y^2}{xy}=\frac{x^4\left(1+\left(\frac{y}{x^2}\right)^2\right)}{x^3\cdot\frac{y}{x^2}}=x\frac{1+\left(\frac{y}{x^2}\right)^2}{\frac{y}{x^2}}.$$

This equation is a particular case of equation (9) for $\alpha = 2$, $f(z) = (1 + z^2)/z$.

IV. Linear Equations. The equation

$$\frac{dy}{dx}+p(x)\,y=f(x),\quad a<x<b, \tag{13}$$

where $p(x)$, $f(x)$ are continuous functions of x on the interval (a, b), is called a first-order *linear* differential equation. The unknown function $y(x)$ and its derivative enter into this equation linearly in the first power.

If $f(x) \equiv 0$, then the equation

$$\frac{dy}{dx}+p(x)\,y=0 \tag{14}$$

is called linear *homogeneous*, and in this connection equation (13) is said to be linear *nonhomogeneous*.

A homogeneous linear equation has the solution $y(x) \equiv 0$. It is an equation with variables separable:

$$\frac{dy}{y}=-p(x)\,dx\quad (y\neq 0),\quad \ln\left|\frac{y}{C}\right|=-\int p(x)\,dx\ (C\neq 0),$$

$$y=C\exp\left(-\int p(x)\,dx\right). \tag{15}$$

If in (15) the constant C is allowed to take on zero value, then formula (15) also yields the solution $y(x) \equiv 0$.

Formula (15) shows that the graph of the solution of a linear homogeneous equation lies above the x-axis if $C > 0$, or below the x-axis if $C < 0$.

We have arrived at formula (15) using the following scheme. We assumed that the function $y = y(x)$ is a solution of differential equation (14), different from zero everywhere on (a, b), and found that it is determined by formula (15) for some C. It should be borne in mind that the integral $\int p(x)\,dx$ designates a certain primitive function of $p(x)$ on the interval (a, b), therefore the solution yielded by for-

mula (15) is also defined on (a, b). It is readily checked that the functions of (15) for any C ($C = 0$ included) are solutions of differential equation (14).

It remains to be seen whether there are solutions of our differential equation intersecting the x-axis. To this end, we may take advantage of Theorem 1 proved in Sec. 1.2. Solving (15) with respect to C, we get

$$C = y \exp\left(\int p(x)\, dx\right).$$

It is easily verified that the right-hand side of this equality is a function of (x, y) having continuous partial derivatives in the strip $\{-\infty < y < \infty,\ a < x < b\}$, as also the fact that, differentiating this equality with respect to x (setting $y = y(x)$), we obtain differential equation (14). Then, by Theorem 1 of Sec. 1.2, formula (15) contains all the solutions of equation (14). Hence, *linear equation* (14) *has no solutions intersecting the x-axis.*

Equation (13) is usually solved by *Bernoulli's* method* which consists in the following. We look for a solution in the form of the product of two functions

$$y(x) = u(x) \cdot v(x).$$

We have $y' = uv' + u'v$. Substituting the values of y and y' into (13), we obtain $uv' + u'v + p(x)\, uv = f(x)$ or $u(v' + p(x)\, v) + u'v = f(x)$.

Let us choose the function v so that $v' + p(x)\, v = 0$. And so we have a homogeneous linear equation with respect to $v(x)$, consequently, by (15), we may set $v = \exp \times$ $\times \left(-\int p(x)\, dx\right)$. For such function v we obtain $u'v = f(x)$, whence

$$du = \frac{f(x)}{v(x)}\, dx,$$

$$u = \int \frac{f(x)}{v(x)}\, dx + C = \int f(x) \exp\left(\int p(x)\, dx\right) dx + C.$$

* Bernoulli, James (1654-1705). Outstanding Swiss mathematician and physicist.

Consequently, a general, i.e. an arbitrary, solution of equation (13) is written in the form

$$y = u \cdot v = C \exp\left(-\int p(x)\,dx\right) + \exp\left(-\int p\,dx\right)\int f(x)\exp\left(\int p\,dx\right)dx, \qquad (16)$$

where C is an arbitrary constant.

Formula (16) shows that the *general solution of a linear nonhomogeneous equation is equal to the sum of the solution of the corresponding homogeneous equation and particular solution of the nonhomogeneous equation* (obtained from (16) for $C = 0$).

We recommend that formula (16) be not used formally and the entire procedure be repeated in each individual example.

Example 6. Solve the equation $y' - y = \sin x$.

Here $p(x) = -1$, $f(x) = \sin x$. We put $y = u \cdot v$,

$$y' = u'v + u \cdot v', \quad uv' + u'v - uv = \sin x,$$

$$u(v' - v) + u'v = \sin x, \quad v' - v = 0,$$

$$v = \exp\int dx = e^x, \; u'e^x = \sin x, \; u' = e^{-x}\sin x,$$

$$u = \int e^{-x}\sin x\,dx + C; \quad y = e^x\int e^{-x}\sin x\,dx + Ce^x.$$

Integrating by parts, we find that

$$y = Ce^x - \frac{1}{2}(\cos x + \sin x).$$

Note. Equation (13) can also be solved using the *method of variation of an arbitrary variable.* If C is constant, then formula (15) yields the solution of a homogeneous equation. Let us regard C as a function of x and choose it so that the expression $y = C(x)\exp\left(-\int p(x)\,dx\right)$ represents the solution of (13). But this is the same Bernoulli's method for $u = C(x)$, $v = \exp\left(-\int p(x)dx\right)$.

V. The Bernoulli Equation:

$$y' + p(x)\,y = y^{\alpha} f(x), \qquad (17)$$

where α is any real number.

If α is equal either to zero or unity, then we obtain linear differential equation. And if $\alpha \neq 0, 1$, then the substitution $z = y^{1-\alpha}$ leads us once again to a linear equation with respect to the function $z(x)$.

We can solve the Bernoulli equation at once by Bernoulli's method, setting $y = u(x) \cdot v(x)$. Note that for $\alpha > 0$ the function $y(x) \equiv 0$ is a solution of the Bernoulli equation.

Sec. 1.4. Existence Theorem on a Solution of a First-order Differential Equation

The class of differential equations which we can effectively solve is quite narrow. For instance, it turns out that the solution of the following differential equation

$$\frac{dy}{dx} = x^2 + y^2,$$

which is so simple at first sight, cannot be reduced even to quadratures (integrals). Therefore, in most cases differential equations are solved approximately.

But prior to using a certain approximate method, we must know whether a solution of a given differential equation exists. It is also of great importance to know beforehand whether it is unique.

Below, we formulate the conditions which guarantee the existence and uniqueness of the solution of the first-order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

for the initial condition

$$y(x_0) = y_0. \tag{2}$$

The following theorem takes place.

Theorem 1. *Let a function $f(x, y)$ be continuous in the rectangle*

$$D = \{x_0 - a \leqslant x \leqslant x_0 + a, \quad y_0 - b \leqslant y \leqslant y_0 + b\}$$

and have in it a bounded derivative $\frac{\partial f}{\partial y}$, satisfying the inequality

$$\left|\frac{\partial f}{\partial y}\right| \leqslant N. \tag{3}$$

Then in the interval $\sigma = [x_0 - \delta,\ x_0 + \delta]$, *where*

$$\delta < \min\left\{a,\ \frac{1}{N},\ \frac{b}{M}\right\},\quad M = \max_{(x,\ y)\in D} |\ f(x,\ y)\ |, \tag{4}$$

there exists a unique solution of equation (1) *satisfying the initial condition* (2).

Here, the following inequality is fulfilled:

$$|\ y\ (x) - y_0\ | \leqslant b,\quad \forall x \in \sigma.$$

The solution $y\ (x)$ *is continuously differentiable on* σ. *And if* $f\ (x,\ y)$ *actually has continuous partial derivatives of order* p

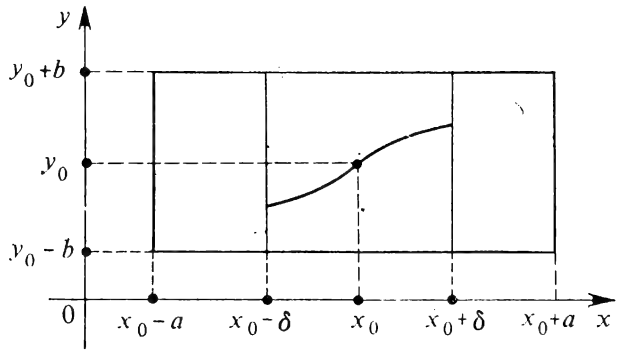

Fig. 7.

with respect to x *and* y, *then* $y\ (x)$ *has on* σ *continuous derivatives up to order* $p + 1$ *included.*

Figure 7 represents a rectangle D in the xy-plane and the rectangle

$$D_1 = \{x_0 - \delta < x \leqslant x_0 + \delta,\ \ y_0 - b \leqslant y \leqslant y_0 + b\}$$

which belongs to the rectangle D.

The theorem states that if the function $f\ (x,\ y)$ is continuous in the rectangle D and has a bounded partial derivative $\frac{\partial f}{\partial y}$ satisfying inequality (3), then through this point $(x_0,\ y_0)$ there passes a unique integral curve $y = y\ (x)$ defined for all values $x \in [x_0 - \delta,\ x_0 + \delta]$. This curve belongs entirely to the rectangle D_1. The number δ satisfies relationships (4).

We should like to underline that Theorem 1 guarantees

the existence of a definite interval

$$\sigma = [x_0 - \delta, \ x_0 + \delta],$$

on which there exists a fortiori the solution

$$y = y(x)$$

of equation (1) passing through the point (x_0, y_0).

Suppose we need to find approximately this solution, then with the indicated information at our disposal we would organize the finding of an approximate solution just on this interval σ, since one cannot warrant that the indicated solution is defined outside σ.

Theorem 1 will be proved in Sec. 1.6 and now we shall consider an example.

Example. The equation

$$\frac{dy}{dx} = -y^2 \tag{5}$$

is a particular case of differential equation (1).

Its right-hand side is independent of x. In this case the function $f(x, y)$ is equal to $-y^2$ for any x.

Since the function $-y^2$ is continuous together with its derivative with respect to y for any y, the function $f(x, y)$ defined by it is continuous together with its partial derivative $\frac{\partial f}{\partial y}$, in the entire xy-plane. Therefore, without solving equation (5), we can conclude on the basis of the existence theorem that through any point (x_0, y_0) there passes a unique integral curve of equation (5).

Suppose $x_0 = 3$, $y_0 = 1$. Let there be given an arbitrary rectangle

$$D = \{3 - a \leqslant x \leqslant 3 + a, \ 1 - b \leqslant y \leqslant 1 + b\}$$
$$(0 < a, \ b).$$

For it

$$M = \max_{(x, y) \in D} |f(x, y)| = \max_{y \in [1-b, \ 1+b]} |-y^2| = \max_{1-b \leqslant y \leqslant 1+b} y^2$$
$$= (1 + b)^2;$$

$$N = \max_{(x, y) \in D} \left| \frac{\partial f}{\partial y} \right| = \max_{1-b \leqslant y \leqslant 1+b} |-2y| = 2(1 + b).$$

Consequently,

$$\delta < \min\left\{a, \frac{1}{2(1+b)}, \frac{b}{(1+b)^2}\right\} < \frac{1}{2}. \tag{6}$$

Equation (5) is solved easily. Its general integral in the upper half-plane ($y > 0$) and in the lower half-plane ($y < 0$) is defined by the equality

$$y = \frac{1}{x - C}. \tag{7}$$

There is one more solution $y \equiv 0$, but we shall not be interested in it.

From solutions (7) we choose the one which passes through the point (3, 1). Obviously, this is the solution $y = \frac{1}{x-2}$

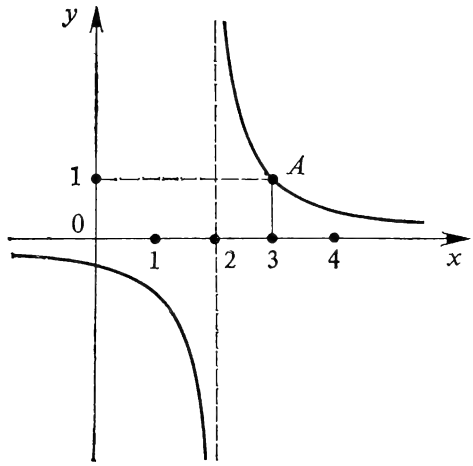

Fig. 8.

Its graph is shown in Fig. 8. We see that the integral curve of equation (5) passing through the point $A = (3, 1)$ goes to infinity as $x \to 2$.

The greater interval with centre at the point $x = 3$ on which our integral curve is defined is the interval (2, 4). Relationship (6) obtained from the general existence theorem, yields a somewhat smaller interval.

The existence theorem on a solution of a differential equation will be proved proceeding from the general considerations pertaining to the theory of metric spaces. The next section is dedicated to the theory.

Sec. 1.5. Metric Space

Let M be a set of members x, y, z, . . . of an arbitrary nature.

The set M is said to be a metric space if to any pair of its members there is associated a nonnegative real number $\rho(x, y)$ called their distance from x to y, which satisfies the following properties (*axioms of distance*):

(1) $\rho(x, y) = 0$, if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$;

(3) $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$, $\forall x, y, z \in M$.

Axiom (3) is usually called the *triangle inequality*. The function $\rho(x, y)$ of two arguments x, y will also be called a *metric* on M.

As is easily seen, the n-dimensional space R_n with the metric

$$\rho(x, y) = |x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ is a metric space.

The set $C[a, b]$ of all continuous functions defined on $[a, b]$ will be a metric space if the metric is introduced by the formula

$$\rho(f, g) = \max_{a \leqslant t \leqslant b} |f(t) - g(t)|. \tag{1}$$

The axioms of distance are readily verified.

Henceforth, the expression $\{x^n\}$ will denote some sequence of elements $x^n \in M$ $(n = 1, 2, \ldots)$. Thus, x^n denotes the element having the number n, but not the power of the element x.

The element $x^0 \in M$ is the limit of $\{x^n\}$, if

$$\rho(x^n, x^0) \to 0, \quad n \to \infty.$$

In this case the sequence $\{x^n\}$ is said *convergent*.

The sequence $\{x^n\}$ is called *fundamental* if $\forall \varepsilon > 0$ $\exists N$ such that

$$\rho(x^m, x^n) < \varepsilon$$

for $m, n > N$.

If the sequence $\{x^n\}$ converges to $x^0 \in M$, then it is fundamental. Indeed, from the convergence of $\{x^n\}$ to x^0 it

follows that for any $\varepsilon > 0$ there is N such that $\rho(x^n, x^0) < \varepsilon$, $\forall n > N$. Therefore, on the basis of the triangle inequality

$$\rho(x^n, x^m) \leqslant \rho(x^n, x^0) + \rho(x^0, x^m) < \varepsilon + \varepsilon = 2\varepsilon$$

for $m, n > N$.

The converse is not always true. For instance, if M $(0, 1)$ is the interval $0 < x < 1$ and $\rho(x, y) = |x - y|$, then $\{1/n\}$ is a fundamental sequence. But it does not converge to an element (point) of space M (it is convergent to zero which does not belong to M).

A metric space M is said to be *complete* if and only if any fundamental sequence converges to a point of this space.

We know that one-dimensional space R_1 (of numbers) is complete (Cauchy's criterion!). It is possible to prove that the space R_n is also complete for $\forall n \geqslant 1$.

Theorem 1. *The space $C[a, b]$ is complete.*

Proof. Let the elements (points) of this space $\{f_n(t)\}$ form a fundamental sequence in the sense of metric (1): for any $\varepsilon > 0$ $\exists N$ such that

$$\rho(f_n, f_m) = \max_{a \leqslant t \leqslant b} |f_n(t) - f_m(t)| < \varepsilon \tag{2}$$

for $m, n > N$.

It follows from (2) that with $t \in [a, b]$ fixed

$$|f_n(t) - f_m(t)| < \varepsilon \quad (n, m > N). \tag{3}$$

The last inequality means that the number sequence $\{f_n(t)\}$ is fundamental, therefore, by Cauchy's criterion, it converges to some real number which will be denoted by $f(t)$:

$$\lim_{n \to \infty} f_n(t) = f(t), \quad \forall t \in [a, b]. \tag{4}$$

Passing to the limit in inequality (3) as $m \to \infty$, we obtain

$$|f_n(t) - f(t)| \leqslant \varepsilon, \quad n > N, \quad \forall t \in [a, b]. \tag{5}$$

Hence

$$\sup_{a \leqslant t \leqslant b} |f_n(t) - f(t)| \leqslant \varepsilon, \quad n > N. \tag{6}$$

This shows that the sequence $\{f_n(t)\}$ converges *uniformly* to $f(t)$ on $[a, b]$ and, since the functions $f_n(t)$ are continuous

on $[a, b]$, the limit function $f(t)$ is also continuous on $[a, b]$*, i.e. $f(t) \in C[a, b]$.

The theorem has been proved.

Note. In inequality (6) the symbol 'sup' can now be replaced by 'max'.

Let in a complete metric space M there be given an operator (function) mapping M into itself

$$z = Fx \quad (x \in M, \ z \in M).$$

Operator F will be called *contractive* if

$$\rho(Fx, Fy) \leqslant \alpha\rho(x, y), \quad \forall x, y \in M, \ 0 \leqslant \alpha < 1,$$

where the number α is independent of x, y.

A point $x \in M$ is said to be the *fixed point* of operator F if $Fx = x$.

Operator F will be called *continuous* at point x^0 if

$$\lim_{x^n \to x^0} Fx^n = Fx^0$$

(i.e. $\rho(Fx^n, Fx^0) \to 0$, $n \to \infty$, $\forall x^n \to x^0$).

It is easily seen that a contractive operator is always continuous at any point $x^0 \in M$

$$\rho(Fx^n, Fx^0) \leqslant \alpha\rho(x^n, x^0) \to 0, \quad n \to \infty.$$

Theorem 2. *If a contractive operator F maps a complete metric space M into itself, then there exists a unique fixed point of this space.*

This theorem is known as the *principle of contracted maps.*

Proof. Let us prove that it is impossible for two fixed points to exist. Let x^1, x^2 be two fixed points: $Fx^1 = x^1$, $Fx^2 = x^2$. Then

$$\rho(x^1, x^2) = \rho(Fx^1, Fx^2) \leqslant \alpha\rho(x^1, x^2) \quad (\alpha < 1). \qquad (7)$$

Assuming that $\rho(x^1, x^2) > 0$, we obtain from (7) that $\alpha \geqslant 1$ which is impossible. Hence $\rho(x^1, x^2) = 0$ and $x^1 = x^2$.

Now we pass over to the proof of the existence of a fixed point.

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* See our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 9.8, Theorem 2, Mir Publishers, Moscow, 1982.

Let x^0 be any point of the space M and let us form a sequence of elements

$$x^0,\ x^1 = Fx^0,\ x^2 = Fx^1,\ \ldots,\ x^n = Fx^{n-1},\ \ldots$$

This sequence will be called *iterated*, generated by the operator F. Let us show that this sequence is fundamental. We have

$$\begin{aligned}\rho(x^n,\ x^{n-1}) = \rho(Fx^{n-1},\ Fx^{n-2}) &\leqslant \alpha\rho(x^{n-1},\ x^{n-2})\\ &\leqslant \alpha^2\rho(x^{n-2},\ x^{n-3})\\ \ldots &\leqslant \alpha^{n-1}\rho(x^1,\ x^0)\quad (n = 1,\ 2,\ \ldots).\end{aligned}\tag{8}$$

Then, on the basis of the triangle inequality and (8), we obtain $(n > m)$

$$\begin{aligned}\rho(x^n,\ x^m) &\leqslant \rho(x^n,\ x^{n-1}) + \rho(x^{n-1},\ x^{n-2}) + \ldots + \rho(x^{m+1},\ x^m)\\ &\leqslant [\alpha^{n-1} + \alpha^{n-2} + \ldots + \alpha^m]\,\rho(x^1,\ x^0).\end{aligned}$$

Since, by the hypothesis $0 \leqslant \alpha < 1$, we get

$$\begin{aligned}\rho(x^n,\ x^m) &\leqslant [\alpha^m + \alpha^{m+1} + \ldots]\,\rho(x^1,\ x^0)\\ &= \frac{\alpha^m}{1-\alpha}\,\rho(x^1,\ x^0) \to 0\end{aligned}$$

as

$m \to \infty$, since $\rho(x^1,\ x^0)$ is a finite number.

Hence, the sequence $\{x^n\}$ is fundamental and, since the space M is complete, this sequence converges to some element $\bar{x}$ of this space

$$\lim_{n\to\infty} x^n = \bar{x} \in M.$$

Let us prove that $\bar{x}$ is a fixed point:

$$\begin{aligned}\rho(F\bar{x},\ \bar{x}) &\leqslant \rho(F\bar{x},\ x^n) + \rho(x^n,\ \bar{x}) = \rho(F\bar{x},\ Fx^{n-1})\\ &\quad + \rho(x^n,\ \bar{x}) \leqslant \alpha\rho(\bar{x},\ x^{n-1}) + \rho(x^n,\ \bar{x}) < \varepsilon\end{aligned}$$

for $n > N = N(\varepsilon)$.

Thus, $\rho(\bar{x},\ F\bar{x}) = 0$, and by the first axiom of distance we conclude that $F\bar{x} = \bar{x}$, i.e. $\bar{x}$ is a fixed point.

The theorem has been proved.

Remark. Using the fact that $\bar{x}$ is a fixed point, we obtain

$$\rho(\bar{x}, x^n) = \rho(F\bar{x}, Fx^{n-1}) \leqslant \alpha\rho(\bar{x}, x^{n-1})$$
$$= \alpha\rho(F\bar{x}, Fx^{n-2}) \leqslant \alpha^2\rho(\bar{x}, x^{n-2}) \leqslant \ldots \leqslant \alpha^n\rho(\bar{x}, x^0). \quad (9)$$

Further

$$\rho(x^n, \bar{x}) \leqslant \rho(x^n, x^{n+1}) + \rho(x^{n+1}, \bar{x})$$
$$\leqslant \alpha\rho(x^{n-1}, x^n) + \alpha\rho(x^n, \bar{x}),$$

whence

$$\rho(x^n, \bar{x}) \leqslant \frac{\alpha}{1-\alpha}\rho(x^{n-1}, x^n) \quad (0 \leqslant \alpha < 1). \quad (10)$$

Formulas (9) and (10) show that x^n is an approximate value of the fixed point with an error not exceeding $\alpha^n\rho(\bar{x}, x^0)$ and $\frac{\alpha}{1-\alpha}\rho(x^{n-1}, x^n)$.

Draw your attention to formula (10) which gives an estimate of the distance between x^n and $\bar{x}$ in terms of the distance between the two neighbouring points x^n and x^{n-1}. Taking x^n for an approximate value of $\bar{x}$, we guarantee that the approximation error is less than the right-hand member of (10).

Example. Let the function $f(x)$ have a root (zero) on $[a, b]$. We shall assume that f has derivatives of the first and second orders and that $f'(x) \neq 0$ on $[a, b]$, i.e. $f(x)$ is monotone on this interval. This implies that there is one root of the function f on $[a, b]$.

Write an auxiliary function

$$F(x) = x + k(x) \cdot f(x),$$

where $k(x)$ is a certain continuously differentiable function different from zero. It is clear that the fixed point $\bar{x}$ of the function F is a zero of f and conversely.

Therefore, if the function F maps $[a, b]$ into $[a, b]$ and is contractive on $[a, b]$, then the iterated sequence $x_n = F(x_{n-1})$ converges to the fixed point F (i.e. to a root of f), and x_n may be taken for an approximate value of the root. For different $k(x)$ we shall obtain different approximate methods for computing the root of the function $f(x)$.

Let us consider a concrete function $f(x) = x^3 + x - 1$ and compute approximately the root of this function to

within 0.01. We have $f(0) = -1$, $f(1) = 1$, $f'(x) = 3x^2 + 1 > 0$. Consequently, there is only one root of $f(x)$ on $[0, 1]$.

Setting $k(x) = -1/f'(x) < 0$, we obtain

$$F(x) = x - \frac{f(x)}{f'(x)} = \frac{2x^3+1}{3x^2+1}.$$

Let us find out whether this function is contractive. By Lagrange's theorem, we get

$$\rho(F(x), F(y)) = |F(x) - F(y)| = |F'(\xi)| \cdot |x - y| \leqslant \alpha\rho(x, y),$$

where

$$\alpha = \max_x |F'(x)| = \max_x \left|\frac{f(x)f''(x)}{(f'(x))^2}\right| = \max_{x]} \left|\frac{(x^3+x-1)\,6x}{(3x^2+1)^2}\right|.$$

For the interval $[1/2, 3/4] \subset [0, 1]$

$$\alpha = \max_{1/2 \leqslant x \leqslant 3/4} |F'(x)| = F'(1/2) = \frac{18}{49} < 1$$

and the values of the function F do not exceed the bounds of $[1/2, 3/4]$:

$$\frac{1}{2} \leqslant \frac{2x^3+1}{3x^2+1} \leqslant \frac{3}{4}, \quad x \in \left[\frac{1}{2}, \frac{3}{4}\right].$$

Thus, on $[1/2, 3/4]$ the function $F(x)$ is contractive.

Let $x_0 = 11/16 \in [1/2, 3/4]$, then $x_1 = F(x_0) = 3379/4952$ and

$$\frac{\alpha}{1-\alpha}\rho(x_1, x_0) = \frac{\alpha}{1-\alpha}|x_1 - x_0| \leqslant \frac{18}{31}\left|\frac{3379}{4952} - \frac{11}{16}\right| = \frac{18 \times 51}{31 \times 2 \times 4952} < \frac{1}{300} < 0.01.$$

On the basis of (10), we can take x_1 for an approximate value of the root of the function $f(x)$ with an accuracy to 0.01 (actually to 0.003).

Remark. The method for approximate computation of the root of the function $f(x)$ ($x_n \approx x$) for $k(x) = -1/f'(x)$ is called *Newton's method of approximation* or the *method of tangents*. The elements of the iterated sequence $\{x_n\}$ can be obtained from geometrical considerations.

If $x_{n-1} \in [a, b]$ is already determined, then, in order to obtain x_n, we draw a tangent line at the point $(x_{n-1}, f(x_{n-1}))$

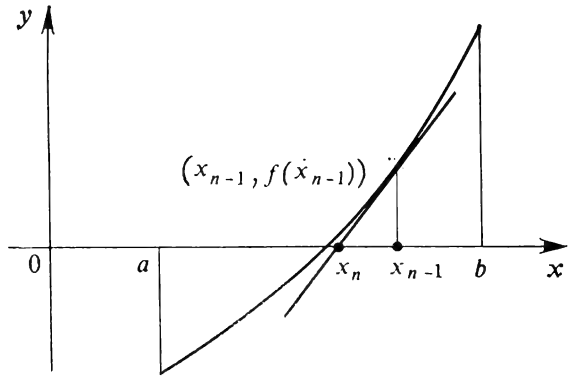

Fig. 9.

of the graph of the function f. The point of intersection of this tangent with the x-axis is taken for x_n (Fig. 9). The equation of the tangent has the form

$$Y - f(x_{n-1}) = f'(x_{n-1})(x - x_{n-1}).$$

Setting $Y = 0$ in this equality, we find the solution $x = x_n$, where

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad (n = 1, 2, \ldots).$$

Thus, the numbers x_n are elements of the iterated sequence for the function $F(x) = x - (f(x)/f'(x))$.

Problem. The function $F(x) = x^2$ maps R_1 into itself and has two fixed points $x = 0$ and $x = 1$, Why?

Sec. 1.6. Proving the Existence Theorem on a Solution of a First-order Differential Equation

Let there be given an equation

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

where the function $f(x, y)$ is continuous in the rectangle

$$D = \{x_0 - a \leqslant x \leqslant x_0 + a,\ y_0 - b \leqslant y \leqslant y_0 + b\}$$

and has a bounded partial derivative $\frac{\partial f}{\partial y}$, satisfying the inequality $\left|\frac{\partial f}{\partial y}\right| \leqslant N$.

We have to prove that on the interval $\sigma = [x_0 - \delta, x_0 + \delta]$ there exists a unique solution $y = y(x)$ of differential equation (1) satisfying the initial condition

$$y(x_0) = y_0, \tag{2}$$

where

$$\delta < \min\left\{a, \frac{1}{N}, \frac{b}{M}\right\}, \quad M = \max_{(x, y) \in D} |f(x, y)|, \tag{3}$$

$y(x)$ being continuously differentiable on σ. Differential equation (1) with condition (2) is equivalent to the following integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt. \tag{4}$$

Indeed, let the continuous function $y(x)$ be a solution of (4), then, differentiating identity (4), we obtain

$$\frac{dy}{dx} = f(x, y(x)) \quad \text{and, obviously,} \quad y(x_0) = y_0.$$

Thus, the function $y(x)$ satisfies the equation (1) with condition (2).

Conversely, let $y(x)$ be a solution of (1):

$$\frac{dy(t)}{dt} = f(t, y(t)), \quad x_0 \leqslant t \leqslant x; \quad y(x_0) = y_0.$$

Then, integrating this identity between x_0 and x, we obtain

$$\int_{x_0}^{x} \frac{dy(t)}{dt}\, dt = \int_{x_0}^{x} f(t, y(t))\, dt$$

or

$$y(x) - y_0 = \int_{x_0}^{x} f(t, y(t))\, dt,$$

i.e. $y(x)$ is a solution of (4).

In what follows we shall investigate equation (4).

Let us denote by $\mathfrak{M}$ the set of continuous functions $y = y(x)$ defined on the interval $\sigma = [x_0 - \delta, x_0 + \delta]$ and satisfying the inequality $|y(x) - y_0| \leqslant b$ on it.

Let us introduce into $\mathfrak{M}$ the distance

$$\rho(y, z) = \max_{x \in \sigma} |y(x) - z(x)|, \quad y, z \in \mathfrak{M}.$$

Hence, $\mathfrak{M}$ is a metric space. This is a complete space. Indeed, if the sequence of functions $y_n \in \mathfrak{M}$ satisfies Cauchy's condition in the sense of the metric introduced (i.e. if it is a fundamental sequence), then, as we know, this sequence converges uniformly on the interval σ to a certain function $y = y(x)$, which is continuous on this interval (see Sec. 1.5).

For the functions $y_n = y_n(x)$ the inequality is fulfilled

$$|y_n(x) - y_0| \leqslant b, \quad x \in \sigma \quad (n = 1, 2, \ldots),$$

which is retained on passing to the limit as $n \to \infty$

$$|y(x) - y_0| \leqslant b.$$

But then $y = y(x) \in \mathfrak{M}$, which shows that $\mathfrak{M}$ is a complete space.

The equality

$$z(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt, \quad y(x_0) = y_0, \tag{5}$$

associates a certain function $z = z(x) \in \mathfrak{M}$ to each function $y = y(x) \in \mathfrak{M}$. Indeed, if $y \in \mathfrak{M}$, then $y = y(t)$ is a continuous function whose graph belongs to the rectangle

$$D_1 = \{x_0 - \delta \leqslant x \leqslant x_0 + \delta, \ y_0 - b \leqslant y \leqslant y_0 + b\},$$

therefore, by virtue of the continuity of $f(x, y)$ on D_1, the right-hand member of equality (5) is a continuous function of x, i.e. $z = z(x)$ is a continuous function on σ. Further

$$|z(x) - y_0| = \left| \int_{x_0}^{x} f(t, y(t))\, dt \right| \leqslant M |x - x_0| \leqslant M\delta < M \cdot \frac{b}{M} = b,$$

which shows that $z \in \mathfrak{M}$.

And so we may assert that equality (5) defines the operator

$$z = Fy, \quad y \in \mathfrak{M}, \quad z \in \mathfrak{M},$$

which maps the complete space $\mathfrak{M}$ into the complete space $\mathfrak{M}$. This is a contractive operator, since if

$$z_1 = Fy_1, \; z_2 = Fy_2, \quad y_1, \; y_2 \in \mathfrak{M}$$

then

$$| z_1(x) - z_2(x) | = \left| \int_{x_0}^{x} [f(t, \; y_1(t)) - f(t, \; y_2(t))] \, dt \right|$$

$$= \left| \int_{x_0}^{x} [y_1(t) - y_2(t)] \, f_y'(t, \; \lambda(t)) \, dt \right| \leqslant \left| \int_{x_0}^{x} \rho(y_1, \; y_2) \, N \, dt \right|$$

$$\leqslant \rho(y_1, \; y_2) \, \delta N = \alpha \rho(y_1, \; y_2), \tag{6}$$

where the number $\alpha = \delta N$ satisfies the inequality $0 \leqslant \leqslant \alpha < 1$, since, by the hypothesis, $\delta < 1/N$. It follows from (6) that

$$\rho(z_1, \; z_2) = \max_{x \in \sigma} | z_1(x) - z_2(x) | \leqslant \alpha \rho(y_1, \; y_2).$$

But then, as we know, in $\mathfrak{M}$ (see the preceding section) there exists a unique function $y = y(x) \in \mathfrak{M}$, for which

$$y = Fy,$$

or, in other words, which satisfies equation (4), and, consequently equation (1) and condition (2).

Using the iteration method, we can obtain an approximate solution of equation (1):

$$y_n(x) = Fy_{n-1}(x) = y_0 + \int_{x_0}^{x} f(t, \; y_{n-1}(t)) \, dt \quad (n = 1, \; 2, \; \ldots), \tag{7}$$

where

$$y_0(x) \equiv y_0 \in \mathfrak{M}.$$

On the basis of formula (10) derived in the preceding section, the estimate of approximation has the form

$$| y(x) - y_n(x) | \leqslant \frac{N\delta}{1 - N\delta} \max_{x \in \sigma} | y_n(x) - y_{n-1}(x) |.$$

There are other approximate methods for solving Cauchy's problem. For instance, Euler's method which is very simple (see Sec. 1.7)

It remains to prove that if $f(x, y)$ has continuous derivatives with respect to x and y up to pth order in D, then the indicated solution $y(x)$ of equation (1) has continuous derivatives with respect to x up to order $p+1$ on σ.

Indeed, there takes place the following identity:

$$y'(x) = f(x, y(x)), \quad x \in \sigma. \tag{8}$$

Since the function $y(x)$ satisfies differential equation (1), it has a derivative with respect to x everywhere on σ and, therefore, it is continuous. Further, by the hypothesis, $f(x, y)$ is continuous with respect to x and y in D, therefore the right-hand member of (8) is continuous with respect to x on σ. Hence, $y'(x)$ is also continuous on σ.

If $p \geqslant 1$, then the right-hand member of (8) has a continuous derivative with respect to the variable x, consequently, the left-hand member of the identity also has a continuous derivative with respect to x. Hence the function $y(x)$ has a continuous derivative of the second order. From identity (8) we find

$$y''(x) = f'_x(x, y(x)) + f'_y(x, y(x))\, y'(x). \tag{9}$$

Applying to identity (9) the same arguments as above, we find that for $p \geqslant 2$ the function $y(x)$ has a third-order derivative on σ and so forth.

Sec. 1.7. Euler's * Method for Approximate Solution of First-order Differential Equations

Let there be given a differential equation

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

We shall assume that the function $f(x, y)$ satisfies the conditions of the existence theorem in the neighbourhood of the point (x_0, y_0). By this theorem, there are an interval

* Euler, Leonhard (1707-1783). Great mathematician, Member of Russian Academy of Sciences, a Swiss by origin.

$[x_0 - \delta, x_0 + \delta]$ and a unique solution $y = y(x)$ of equation (1) defined on this interval and satisfying the condition $y(x_0) = y_0$.

For the number δ the theorem provides the upper bound

$$\delta < (a, 1/N, b/M).$$

Euler's method makes it possible to express approximately the indicated function theoretically with any preassigned accuracy.

Let it be required to approximate $y(d)$, where, for the sake of definiteness, $x_0 < d < x_0 + \delta$. We separate $[x_0, d]$ into n equal parts by means of the points $x_0, x_1, \ldots, x_n = d$. The length of the interval $[x_i, x_{i+1}]$, $h = x_{i+1} - x_i$, will be called the *computation step*. Approximate values of the solution at points x_i will be denoted by y_i.

On $[x_0, x_1]$, instead of equation (1), we shall consider the equation with the initial condition (Cauchy's problem)

$$Y_n'(x) = f(x_0, y_0) \quad (x_0 \leqslant x \leqslant x_1), \quad Y_n(x_0) = y_0.$$

The solution of this equation has the form

$$Y_n(x) = y_0 + f(x_0, y_0)(x - x_0). \tag{2}$$

This linear function will be just taken for an approximate solution of equation (1) on the interval $[x_0, x_1]$. From the geometrical point of view, this means that we have replaced the sought-for integral curve by a segment of the tangent line to the integral curve at the point (x_0, y_0).

From formula (2) we obtain

$$y_1 = Y_n(x_1) = y_0 + hf(x_0, y_0).$$

We further reason by induction. If the approximate values of the solution $y_1, y_2, \ldots, y_k$ are known, then, instead of equation (1), we consider on $[x_k, x_{k+1}]$ the equation

$$Y_n'(x) = f(x_k, y_k) \quad (x_k \leqslant x \leqslant x_{k+1}), \quad Y_n(x_k) = y_k.$$

The solution of this equation

$$Y_n(x) = y_k + f(x_k, y_k)(x - x_k) \quad (k = 0, 1, \ldots, n - 1) \tag{3}$$

is taken for an approximate solution of equation (1) on $[x_k, x_{k+1}]$.

Setting in (3) $x = x_{k+1}$, we obtain

$$y_{k+1} = Y_n(x_{k+1}) = y_k + hf(x_k, y_k)$$
$$(k = 0, 1, \ldots, n-1). \quad (4)$$

Formulas (4) just define Euler's method.

The function $Y_n(x)$ defined on $[x_0, d]$ with the aid of equalities (3) is called the *Eulerian polygonal line* (Fig. 10).

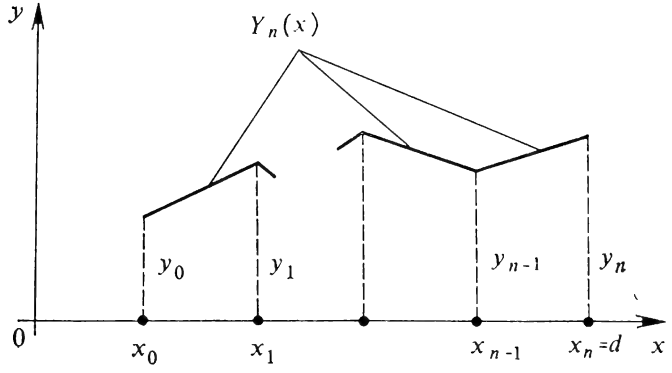

Fig. 10.

It is possible to prove that under the conditions of the existence theorem the sequence of Euler's polygonal lines $\{Y_n(x)\}$ uniformly converges on $[x_0, d]$ to the true solution of the problem as $n \to \infty$.

Sec. 1.8. Equations Not Solved with Respect to Derivative

To solve the differential equation

$$F(x, y, y') = 0, \quad (1)$$

we may first attempt to solve it with respect to y'. If it works well, then we shall obtain one or many differential equations of the form

$$\frac{dy}{dx} = f(x, y). \quad (2)$$

Any solution of each of equations (2) will be a solution of equation (1). And still, one should try to find out whether they exhaust all the solutions of (1).

For instance, to solve the equation

$$(y')^2 - (2x + y)\,y' + 2xy = 0, \tag{3}$$

let us first, by means of identical transformations of its left-hand side, reduce it to the form

$$(y' - 2x)\,(y' - y) = 0. \tag{3'}$$

Consider two first-order differential equations

$$y' = 2x, \quad y' = y.$$

Their general integrals have, respectively, the form

$$y = x^2 + C_1, \quad y = C_2 e^x, \tag{4}$$

where C_1 and C_2 are arbitrary constants. For particular values of C_1 and C_2 functions (4) are particular solutions of equation (3).

But from the indicated particular solutions of the last two equations it is also possible to construct other particular solutions of equation (3). For instance, the function

$$y = \begin{cases} x^2 + 1, & x \leqslant 1, \\ 2e^{x-1}, & x > 1, \end{cases}$$

is a solution of equation (3). This integral curve is made up of two integral curves belonging to different families (4) (Fig. 11).

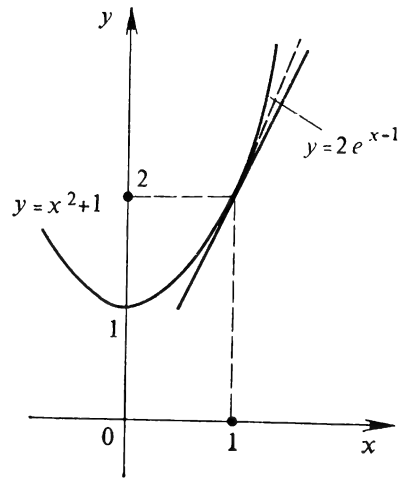

Fig. 11.

Below, we consider two particular forms of differential equation (1) for which it is possible to indicate other ways of their solution.

1°. The left-hand member of equation (1) contains neither x, nor y:

$$F(y') = 0. \tag{5}$$

We shall assume that the function F is continuous and has a finite number of zeros.

Let $y = y(x)$ be a solution of the equation having a continuous derivative. Then $y'(x)$ is equal to one of the roots of equation (5) and will be denoted by k. Hence, $y' = k$, whence $y = kx + C$, where C is a constant and

$$F\left(\frac{y-C}{x}\right) = 0. \tag{6}$$

Conversely, from the fact that for a continuously differentiable function $y(x)$ equality (6) is fulfilled for some constant C it follows that

$$\frac{y-C}{x} = k, \quad \forall x \neq 0,$$

where k is a root of the function F. But then $y = kx + C$, $\forall x$, $y' = k$, and $F(y') = 0$.

We have proved that the *general (any) solution of differential equation* (5) *is defined by equality* (6), *where C is an arbitrary constant.*

2°. The left-hand member of equation (1) does not involve x:

$$F(y, y') = 0. \tag{7}$$

If equation (7) can be solved with respect to y', then $y' = \varphi(y)$ is an equation with variables separable which we know how to solve.

Suppose it is impossible or difficult to solve equation (7) with respect to y', but it is easily solved with respect to y: $y = \varphi(y')$.

Let us introduce into consideration the parameter $p = \frac{dy}{dx}$, then

$$y = \varphi(p), \quad dy = \varphi'(p)\,dp, \quad dx = \frac{dy}{p} = \frac{\varphi'(p)\,dp}{p},$$

whence

$$x = \int \frac{\varphi'(p)\,dp}{p} + C,$$

or

$$x = \psi(p) + C.$$

Now, eliminating the parameter p from the system

$$x = \psi(p) + C, \quad y = \varphi(p), \tag{8}$$

we obtain the general integral $\Phi(x, y, C) = 0$ of differential equation (7).

System (8) can also be regarded as parametric representation of the solution of equation (7). The parameter p can also be introduced in an arbitrary way: $y' = \omega(p)$, but so that equation (7) is solved simpler with respect to y, $y = \varphi(p)$, and that the corresponding integral for defining the function

$$x = \int \frac{\varphi'(p)\,dp}{\omega(p)} + C$$

is found in a simpler way.

Example. Solve the equation $x\sqrt{1 + y'^2} = 2y'$.

By introducing the parameter $p = y'$, we would obtain rather complicated integrals. Here it is better to set $y' = \tan p$ $(-\pi/2 < p < \pi/2)$. Then

$$x = \frac{2\tan p}{\sqrt{1+\tan^2 p}} = 2\sin p,$$

$$dx = 2\cos p\,dp,$$

$$dy = \tan p\,dx = 2\sin p\,dp,$$

$$y = -2\cos p + C.$$

From the system

$$x = 2\sin p, \quad y = -2\cos p + C$$

we obtain

$$x^2 + (y - C)^2 = 4, \qquad (4)$$

i.e. any solution of our differential equation is a solution of equation (4) for a certain constant C. This is a family of circles of radius 2 centred at points $(0, C)$ (Fig. 12). It is possible to prove that equality (4) is the general integral for solutions of the form $y(x)$ and of the form $x(y)$.

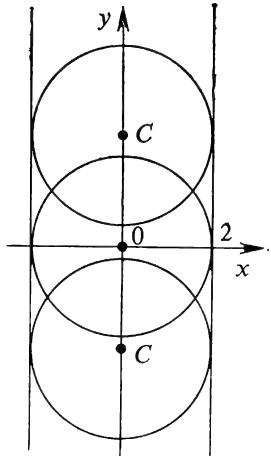

Fig. 12.

In the present case the parameter can be introduced by the formula $y' = \sinh p$.

Remark. A differential equation of the form $F(x, y') = 0$ is considered in a similar way.

Sec. 1.9. Singular Solutions

Let there be a differential equation

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

If in the neighbourhood of point (x_0, y_0) of the xy-plane the conditions of the existence and uniqueness theorem are fulfilled, then through it there passes only one integral curve.

If the conditions of the existence theorem are not observed, then various cases are possible. Through the point (x_0, y_0) there may pass one integral curve, several curves or an infinite number of them, or there is no integral curve to pass through this point. Of particular interest is the case when differential equation (1) has a singular solution.

A solution of a first-order differential equation is called *singular* if the corresponding integral curve possesses the property that through any of its points there passes another integral curve of the given equation which is tangent to the first curve.

It is not a rare case when we have to deal with differential equation of form (1), where the function $f(x, y)$ is continuous on some domain Ω, and its partial derivative $\frac{\partial f}{\partial y}$ is finite and continuous not everywhere in Ω. There are such points in Ω where $\frac{\partial f}{\partial y} = \infty$. At such points, generally speaking, the conditions of existence and uniqueness of the solution of differential equation (1) are violated, and if such points form smooth lines, then the latter can represent singular solutions of the differential equation.

Example 1. Consider the simplest Bernoulli equation $y' = y^\alpha$, $\alpha > 0$, $y \geqslant 0$. Here $f(x, y) = y^\alpha$ is a continuous function in the upper half-plane. For $0 < \alpha < 1$ the function $\frac{\partial f}{\partial y} = \alpha y^{\alpha-1}$ is not bounded in the neighbourhood of $y = 0$. The function $y = 0$ is a solution of the equation. For the initial condition $y(x_0) = 0$ there is one more solution

$$y = [(x - x_0)(1 - \alpha)]^{1/(1-\alpha)},$$

satisfying this equation and passing through the point $(x_0, 0)$. The tangent to this curve at the point $(x_0, 0)$ is, obviously, the x-axis ($y \equiv 0$). Therefore $y \equiv 0$ is a singular solution.

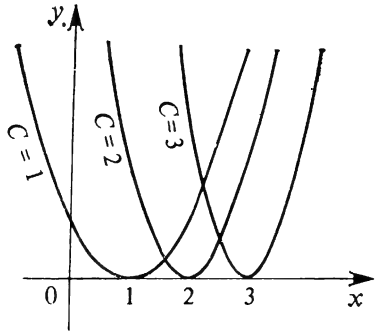

Fig. 13.

Example 2. $y' = y^{\alpha} + 1$, $y \geqslant 0$.

Here $\frac{\partial f}{\partial y} = \alpha y^{\alpha-1}$ and for $0 < \alpha < 1$ this function is not bounded in the neighbourhood of $y = 0$. But $y \equiv 0$ is not a solution of the equation. The solution of the equation, for instance for $\alpha = 1/2$, is defined implicitly by the equality $x + C = 2(\sqrt{y} - \ln(\sqrt{y} + 1))$ $(y \geqslant 0)$, i.e. through each point $(x_0, 0)$ there passes a unique integral curve $x - x_0 = 2(\sqrt{y} - \ln(\sqrt{y} + 1))$.

Example 3. The functions $y = C(x - C)^2$ (Fig. 13) are solutions of the equation $F(x, y, y') \equiv 4xyy' - y'^3 - 8y^2 = 0$ for any C. The function $y \equiv 0$ is a singular solution of the given equation.

Sec. 1.10. The Envelope of a Family of Curves

Let there be given a family of smooth curves Γ_σ defined by the equation

$$\Phi(x, y, \alpha) = 0, \tag{1}$$

where α is an arbitrary constant (parameter) and $\Phi(x, y, \alpha)$ is a function continuously differentiable in some domain of points (x, y, α).

A curve E is called the *envelope* of the family of curves (1) if it is tangent to (has a common tangent with) every curve Γ_α of the family, thus consisting entirely of these points of tangency.

Here is a more precise definition: the *envelope* E of a family of curves Γ_α depending on a parameter α, where $a < \alpha < b$,

is a smooth curve

$$\left.\begin{aligned} x &= x(\alpha), \\ y &= y(\alpha), \end{aligned}\right\} \quad a < \alpha < b, \tag{2}$$

that is tangent to the corresponding curve Γ_α for any value of the parameter α.

The curves Γ_α will be called *enveloped* (Fig. 14).

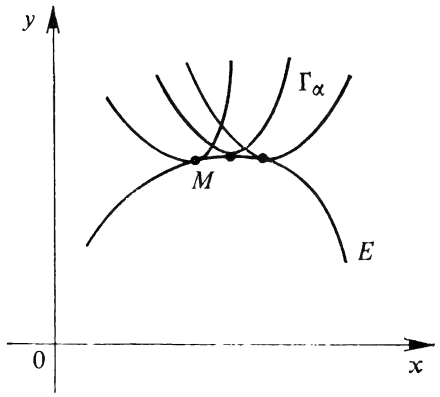

Fig. 14.

Let us find the equation of the envelope. We assign the value α. To the latter there corresponds on Γ_α the point $x = x(\alpha)$, $y = y(\alpha)$ belonging both to Γ_α and E at which E and Γ_α have a common tangent.

Obviously, there takes place the identity

$$\Phi(x(\alpha), y(\alpha), \alpha) \equiv 0, \quad a < \alpha < b.$$

Let us differentiate it with respect to α:

$$\Phi'_x x'(\alpha) + \Phi'_y y'(\alpha) + \Phi'_\alpha \equiv 0, \quad a < \alpha < b.$$

The vector $(x'(\alpha), y'(\alpha))$ is directed along the tangent to E which coincides with the tangent to Γ_α at the point $(x(\alpha), y(\alpha))$. But then (see our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 8.16).

$$\Phi'_x x'(\alpha) + \Phi'_y y'(\alpha) = 0.$$

Consequently,

$$\Phi'_\alpha(x(\alpha), y(a), \alpha) = 0,$$

i.e.

$$\Phi'_\alpha(x, y, \alpha) = 0 \tag{3}$$

at the point (x, y), where E touches Γ_α. For this point the equality

$$\Phi(x, y, \alpha) = 0 \tag{4}$$

is also fulfilled.

Thus, the equation of the envelope of family (1) is defined by two equations

$$\left.\begin{aligned} &\Phi(x, y, \alpha) = 0, \\ &\frac{\partial \Phi(x, y, \alpha)}{\partial \alpha} = 0. \end{aligned}\right\} \tag{5}$$

Equalities (5) yield a necessary condition of the existence of an envelope, i.e. if family (1) has an envelope, then its equation is given by system (5).

But if we have formed system (5) and solved it, then the solution of this system not necessarily yields the envelope of family (1).

Let now there be given a differential equation $F(x, y, y') = 0$ and its general integral $\Phi(x, y, C) = 0$.

If the family of integral curves $\Phi(x, y, C)$ has an envelope, then, evidently, it is also an integral curve and, consequently, a singular solution.

But if C is formally eliminated from the system

$$\left.\begin{aligned} &\Phi(x, y, C) = 0, \\ &\frac{\partial \Phi(x, y, C)}{\partial C} = 0, \end{aligned}\right\} \tag{6}$$

then in some cases we shall obtain a singular solution.

If the general integral of a differential equation has the form

$$\Psi(x, y) = C,$$

where $\Psi(x, y)$ is a continuously differentiable function, then the family (of all solutions of the differential equation) defined by this integral on the respective domain has no envelope (see Theorem 1, Sec. 1.2).

But if the function $\Psi(x, y)$ is not a continuously differentiable at some points (x, y), then the totality of these points can give an envelope of the family.

Example 1. $y' = y^{2/3}$.

This is Bernoulli's equation. Separating the variables, we obtain $y^{-2/3}\,dy = dx$ $(y \neq 0)$, $3y^{1/3} = x - C$, $27y = (x - C)^3$, $\Phi(x, y, C) \equiv 27y - (x - C)^3 = 0$ which is the general integral, where the function $\Phi(x, y, C)$ is continuously differentiable.

Let us form system (6):

$$\left.\begin{aligned} 27y - (x - C)^3 &= 0, \\ 3(x - C)^2 &= 0. \end{aligned}\right\}$$

Eliminating C, we get $y = 0$. We make sure by a check that $y = 0$ is a solution of the original equation. This is a singular solution (see Example 1, Sec. 1.9 and the Note to Example 3, Sec. 1.2) and the envelope of the family of curves $27y - (x - C)^3 = 0$.

If we consider the general integral in the form $C = x - 3y^{1/3}$, then the function $\Psi(x, y) = x - 3y^{1/3}$ is not continuously differentiable at points $y = 0$ which, as we made sure above, are an envelope of the family of cubical parabolas.

Sec. 1.11. Second-order Differential Equations

The equation

$$F(x, y, y', y'') = 0 \tag{1}$$

is called a *second-order differential equation.*

It is assumed that $F(u, v, w, g)$ is a given continuously differentiable function of points (u, v, w, g) belonging to some domain Ω of the four-dimensional space.

Any function $y = y(x)$ having on a certain interval a continuous second-order derivative and satisfying equation (1) is called a solution of this equation or its integral curve.

Each of them $(y = y(x))$ is, generally speaking, defined on its interval $a < x < b$. Of course, for any x from this interval

$$(x, y(x), y'(x), y''(x)) \in \Omega.$$

Sometimes the sought-for solution is imposed by additional conditions. Of special interest are such conditions which guarantee a unique solution of an equation. These

conditions usually have the form

$$y(x_0) = y_0, \quad y'(x_0) = y_0' \tag{2}$$

and are called the initial conditions. The problem of finding the solution of equation (1) satisfying the initial conditions (2) is called Cauchy's problem. From the geometrical point of view, conditions (2) mean that from the family of integral curves passing through the point (x_0, y_0) we single out a certain integral curve having a preassigned slope ($\tan \alpha = y'(x_0) = y_0'$, Fig. 15).

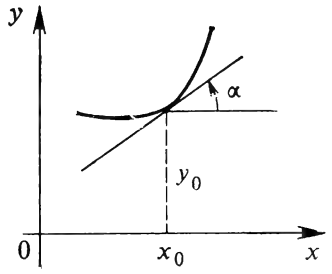

Fig. 15.

Not all the variables x, y, y' may enter into equation (1), but y'' must necessarily enter into it, since otherwise this equation will not be a second-order differential equation.

Examples. $x^2 + y' + y'' = 0$, $y'y'' + 1 = 0$.

Let us solve equation (1) with respect to y''. We shall suppose that this is possible (it is known from the theory of implicit functions that if the function $F(u, v, w, g)$ is equal to zero at some point (u_0, v_0, w_0, g_0), has continuous partial derivatives in the neighbourhood of this point and the partial derivative $\frac{\partial F}{\partial g} \neq 0$ at this point, then the equation $F(u, v, w, g) = 0$ has a unique solution $g = f(u, v, w)$ in a neighbourhood of the indicated point).

Then equation (1) will take the form

$$y'' = f(x, y, y'), \tag{3}$$

where the function $f(u, v, w)$ is defined in some domain ω of the three-dimensional space of points (u, v, w) continuous on it and has continuous partial derivatives. The function f may not depend explicitly on some of the variables x, y, y'. For instance, this takes place for the equations $y'' = \varphi(x)$, $y'' = y' + y$, $y'' = y$, $y'' = y'$.

Let an integral curve $y = y(x)$ pass through a point (x_0, y_0) and let the tangent to the curve at this point have

a slope equal to a given number y_0' (i.e. $y(x_0) = y_0$, $y'(x_0) = y_0'$).

This defines uniquely the second derivative of $y(x)$ at the point x_0 which is equal to

$$y_0'' = f(x_0, y_0, y_0') \quad (y_0'' = y''(x_0)).$$

But there arises the following question: if we assign $x = x_0$ and arbitrary numbers y_0, y_0', then does actually the integral curve $y = y(x)$ of equation (3) exist for which $y(x_0) = y_0$ and $y'(x_0) = y_0'$ and how many are there such integral curves. The following theorem shows that if the function f is sufficiently smooth in the neighbourhood of the point (x_0, y_0, y_0'), then such integral curve exists and is unique.

Theorem. *Let the right-hand member of equation* (3), *regarded as a function of three variables* (x, y, y') *defined in a three-dimensional domain* ω, *be continuous and have continuous partial derivatives* $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial y'}$ *in this domain.*

Then, whatever the point $(x_0, y_0, y_0') \in \omega$ *may be, there exists an interval* (a, b) *and defined on it a twice continuously differentiable function* $y = y(x)$ *satisfying differential equation* (3).

A function possessing the indicated properties is unique.

The function $y(x)$ is said to be a solution (an integral curve) of differential equation (3) satisfying the initial conditions (2). Or it is also said to solve Cauchy's problem for the indicated initial conditions.

Every such solution is conveniently written in the form

$$y(x) = y(x, y_0, y_0'),$$

where y_0, y_0' are the parameters of the solution. They are independent and may be taken arbitrarily as soon as $(x, y_0, y_0') \in \omega$.

If x_0 is fixed, then to every system of numbers $C_1 = y_0$, $C_2 = y_0'$, $(x_0, C_1, C_2) \in \omega$, there corresponds a solution of differential equation (3) which can be written (with x_0 fixed) in the form

$$y = y(x, C_1, C_2),$$

where C_1, C_2 are arbitrary constants (parameters).

Example. Find the integral curve of the equation $y'' + y = 0$, passing through the point (0, 1) whose tangent at this point has the slope $y'(0) = 0$.

It is easily verified that the function $y = C_1 \cos x + C_2 \sin x$ is a solution of the given equation for any constants C_1 and C_2. Further, $y(0) = C_1$, $y'(0) = C_2$. For the initial conditions to be fulfilled, it is necessary to set $C_1 = 1$, $C_2 = 0$. Thus, the required integral curve has the form: $y = \cos x$.

Sec. 1.12. System of Two First-order Differential Equations

In the equation

$$\frac{d^2y}{dx^2} = f(x, y, y') \tag{1}$$

alongside with its solution $y = y(x)$, $x \in (a, b)$, let there be introduced a function $z = z(x)$, where y' is set to be equal to z. Then it will be equivalent to the following system of two first-order differential equations:

$$\left.\begin{aligned} y' &= z, \\ z' &= f(x, y, z) \end{aligned}\right\} \tag{2}$$

with respect to two unknown functions y and z.

Indeed, let $y(x)$, $x \in (a, b)$, be a solution of differential equation (1). It has a continuous second derivative on (a, b). Then $z(x) = y'(x)$ has a continuous first derivative on (a, b).

Thus, the functions $y(x)$ and $z(x)$ have continuous derivatives and satisfy the system of differential equations (2).

Conversely, if two functions $y(x)$, $z(x)$, $x \in (a, b)$ have continuous derivatives on (a, b) and satisfy system (2), then it follows from the first equation of system (2) that $y(x)$ has a continuous second derivative on (a, b); and, substituting z from the first equation into the second, we obtain that $y(x)$ is a solution of differential equation (1).

System (2) is a particular case of the system

$$\left.\begin{aligned} \frac{dy}{dx} &= \varphi(x, y, z), \\ \frac{dz}{dx} &= \psi(x, y, z) \end{aligned}\right\} \tag{3}$$

with respect to the unknown functions y and z.

This last system is, obviously, a particular case of the system

$$\left.\begin{aligned} F(x, y, z, y', z') &= 0, \\ \Phi(x, y, z, y', z') &= 0, \end{aligned}\right\} \tag{4}$$

where we shall assume that the functions F and Φ are continuous and have continuous partial derivatives with respect to y, y', z, and z' in some domain of points x, y, z, y', z'.

The pair of functions $y(x)$, $z(x)$ is called a *solution of the system of differential equations* (4) if these functions are defined on some interval (a, b) dependent on these functions, have continuous derivatives and satisfy system (4) on (a, b).

Solving equations (4) with respect to y' and z', we obtain a system of form (3) (of course, we suppose that a solution of system (4) with respect to y' and z' is possible which, as is known, is usually connected with the inequality to zero of the Jacobian $\frac{D(F,\Phi)}{D(y', z')}$).

Equations (3) (or (4)) form a system of two first-order differential equations in two unknown functions y and z.

System (3) solved with respect to $\frac{dy}{dx}$, $\frac{dz}{dx}$ is called *normal*.

For a normal system (3) the following existence theorem is valid.

Theorem. *Let two functions* $\varphi(x, y, z)$ and $\psi(x, y, z)$ *be continuous and have continuous partial derivatives with respect to* y *and* z *in a domain* ω *of points* (x, y, z) *and let there be given an arbitrary point* $(x_0, y_0, z_0) \in \omega$.

Then there exists an interval (a, b) *and continuously differentiable functions* $y = y(x)$, $z = z(x)$ *defined on this interval which satisfy system* (3) *and the initial conditions*

$$y(x_0) = y_0, \quad z(x_0) = z_0. \tag{5}$$

The above mentioned functions $y = y(x)$, $z = z(x)$ *are unique.*

And if the functions φ *and* ψ *have continuous partial derivatives of order* p, *then the solutions* $y(x)$ *and* $z(x)$ *are continuously differentiable* $p + 1$ *times on the indicated interval* (a, b).

It was shown above that the solution of second-order equation (1) in one unknown function can be reduced to solution of two first-order equations in two unknown functions (system (2)).

But a general system of first-order differential equations of form (3) is also reduced to a solution of one differential equation of the second order. Indeed, substituting into system (3) some of its solutions $y = y(x)$, $z = z(x)$ for y and z, and differentiating the first equation with respect to x, we obtain

$$\frac{d^2y}{dx^2} = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \cdot \varphi + \frac{\partial \varphi}{\partial z} \cdot \psi \equiv \Phi(x, y, z). \tag{6}$$

Alongside with (6), we shall also consider the first equation of (3)

$$\frac{dy}{dx} = \varphi(x, y, z), \tag{7}$$

with the substitutions $y = y(x)$, $z = z(x)$ made into it.

Let us find z from (7) ($z = \chi(x, y, y')$) and substitute it into (6). We shall then obtain a second-order differential equation

$$\frac{d^2y}{dx^2} = \Phi(x, y, \chi(x, y, y')) \equiv \Lambda(x, y, y') \tag{8}$$

with respect to the function $y = y(x)$ under consideration.

We have proved that if $y(x)$, $z(x)$ are solutions of system (3), then $y(x)$ is a solution of a second-order equation.

Of course, to make these calculations possible, new properties were imposed on the functions φ and ψ, viz.: the continuous differentiability of φ with respect to x, y, z and possibility to solve the first equation of (3) with respect to z.

Sec. 1.13. Differential Equations of the *n*th Order

The equation

$$F(x, y, y', \ldots, y^{(n)}) = 0 \tag{1}$$

is called an *nth order differential equation.*

Here $F(u, v_0, v_1, \ldots, v_n)$ is a function, continuous together with its partial derivatives $\frac{\partial F}{\partial v_0}, \ldots, \frac{\partial F}{\partial v_n}$ in a certain domain Ω of points $(u, v_0, v_1, \ldots, v_n)$ of $(n+2)$-dimensional space.

Solving equation (1) with respect to $y^{(n)}$, we obtain

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}). \tag{2}$$

The following theorem holds true.

Theorem 1 (of existence). *Let the right-hand member* $f(x, y, y' \ldots, y^{(n-1)})$ *of equation* (2) *regarded as a function of* $n+1$ *variables be continuous and have continuous partial derivatives* $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial y'}, \ldots, \frac{\partial f}{\partial y^{(n-1)}}$ *in some domain* Ω *of points* $(x_0, y_0, y_0', \ldots, y_0^{n-1})$.

Then there exist an interval (a, b) *and defined on it an* n *times continuously differentiable function* $y = y(x)$ *satisfying equation* (2) *and the initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)}. \tag{3}$$

The function $y = y(x)$ *possessing the above indicated properties is unique.*

Thus, $y(x)$ *is a solution of equation* (2) *satisfying initial conditions* (3).

If x_0 is fixed, then to each system of numbers

$$C_1 = y_0, \quad C_2 = y_0', \ldots, C_n = y_0^{(n-1)},$$

possessing the property

$$(x_0, C_1, \ldots, C_n) \in \Omega,$$

there will correspond a solution of our differential equation which (with x_0 fixed!) can be written in the form

$$y = y(x, C_1, \ldots, C_n). \tag{4}$$

As a result, we obtain a family of solutions of our differential equation depending on n parameters $C_1, \ldots, C_n$. To each definite system $(C_1, \ldots, C_n)$ of parameters $((x_0, C_1, \ldots, C_n) \in \Omega)$ there corresponds its own solution of the differential equation (with its own interval of definition).

The new functions

$$y_1(x) = y,\ y_2(x) = y',\ \ldots,\ y_n(x) = y^{(n-1)}$$

can be introduced into equation (2). In any case, all of them have a continuous first derivative. Then equation (2) will turn out to be equivalent to the following system of n first-order differential equations:

$$\left.\begin{aligned} &\frac{dy_1}{dx} = y_2, \\ &\frac{dy_2}{dx} = y_3, \\ &\cdots\cdots\cdots\cdots\cdots \\ &\frac{dy_{n-1}}{dx} = y_n, \\ &\frac{dy_n}{dx} = f(x, y_1, y_2, \ldots, y_n). \end{aligned}\right\} \tag{5}$$

System (5) is a particular case of the system

$$\left.\begin{aligned} &\frac{dy_1}{dx} = \varphi_1(x, y_1, \ldots, y_n), \\ &\cdots\cdots\cdots\cdots\cdots \\ &\frac{dy_n}{dx} = \varphi_n(x, y_1, \ldots, y_n), \\ &((x, y_1, \ldots, y_n) \in \Omega), \end{aligned}\right\} \tag{6}$$

of n first-order differential equations in n unknown functions $y_1, \ldots, y_n$.

This is a normal system (solved with respect to the derivatives $\frac{\partial y_j}{\partial x}$ $(j = 1, \ldots, n)$), which is a particular case of the system

$$F_j(x, y_1, \ldots, y_n) = 0 \quad (j = 1, \ldots, n). \tag{7}$$

The following theorem holds.

Theorem 2 (of existence). *Let the functions* $\varphi_j(x, y_1, \ldots, y_n)$ $(j = 1, \ldots, n)$ *be continuous, have continuous partial derivatives of the first order with respect to all the variables, beginning with the second one, in a domain* Ω *of points* $(x, y_1, \ldots, y_n)$, *and let there be given a certain point* $(x_0, y_{10}, \ldots, y_{n0})$ *of this domain.*

Then there exists an interval (a, b) *and defined on it continuously differentiable functions* $y_1(x), \ldots, y_n(x)$ *satisfying system* (6) *and the initial conditions*

$$y_1(x_0) = y_{10}, \ldots, y_n(x_0) = y_{n0}. \tag{8}$$

If the functions $\varphi_j(x, y_1, \ldots, y_n)$ *are* p *times continuously differentiable on* Ω, *then respective solutions of the system* $y_j(x)$ $(j = 1, \ldots, n)$ *have better properties, i.e. they have a continuous derivative of order* $p + 1$.

If x_0 is fixed, then to each system of the numbers

$$C_1 = y_{10}, \ldots, C_n = y_{n0}, \quad (x_0, C_1, \ldots, C_n) \in \Omega$$

there corresponds a solution of system (6) which can be written (with x_0 fixed!) in the form

$$y = y(x, C_1, \ldots, C_n), \tag{9}$$

where $C_1, \ldots, C_n$ are arbitrary constants (parameters).

It was noticed above that the solving of an nth-order equation can be reduced to solving a system of n first-order differential equations in n unknown functions. But the converse is also true: *under certain conditions the solution of system* (6) *can be reduced to solving a certain nth-order differential equation in one unknown function.*

The proof of this converse statement presents a development of the appropriate statement in the case $n = 2$ (see Sec. 1.12).

Example. Reduce the system

$$\frac{dy_1}{dt} = y_1 + y_2 + y_3,$$

$$\frac{dy_2}{dt} = y_2 - 2y_3,$$

$$\frac{dy_3}{dt} = y_1 + y_3$$

to a third-order differential equation.

We shall reduce our system to a differential equation with respect to the function $y_1(t)$. Differentiating the first equation of the given system and taking into consideration two others, we have

$$\frac{d^2y_1}{dt^3} = \frac{dy_1}{dt} + \frac{dy_2}{dt} + \frac{dy_3}{dt} = \frac{dy_1}{dt} + y_1 + y_2 - y_3.$$

Differentiating once again, we obtain

$$\frac{d^3y_1}{dt^3}=\frac{d^2y_1}{dt^2}+\frac{dy_1}{dt}+\frac{dy_2}{dt}-\frac{dy_3}{dt}=\frac{d^2y_1}{dt^2}+\frac{dy_1}{dt}-y_1+y_2-3y_3. \tag{10}$$

From the system of equations

$$\left.\begin{aligned}\frac{dy_1}{dt}&=y_1+y_2+y_3,\\ \frac{d^2y_1}{dt^2}&=\frac{dy_1}{dt}+y_1+y_2-y_3\end{aligned}\right\} \tag{11}$$

we express y_2 and y_3 in terms of y_1 and y_1', y_1'':

$$\left.\begin{aligned}y_2&=\frac{1}{2}(y_1''-2y_1),\\ y_3&=\frac{1}{2}(-y_1''+2y_1').\end{aligned}\right\} \tag{12}$$

Substituting these values of y_2 and y_3 into (10), we get the required third-order equation with respect to the function $y_1(t)$:

$$y_1'''-3y_1''+2y_1'+2y_1=0.$$

Solving this equation, we find the function y_1, and then the functions y_2 and y_3, using formulas (12).

Sec. 1.14. Reducing the Order of a Differential Equation

In many cases we succeed in reducing a differential equation of order n

$$F(x, y, y', \ldots, y^{(n)})=0 \tag{1}$$

to a differential equation of a lower order by introducing a new unknown function. Let us consider some types of equations allowing for reducing the order.

I. Let the left-hand member of equation (1) not contain explicitly the sought-for function, y, i.e. the equation has the form

$$F(x, y', \ldots, y^{(n)})=0. \tag{2}$$

Let us introduce a new function $z(x)=y'(x)$, then $z'=y''$, $\ldots$, $z^{(n-1)}=y^{(n)}$ and equation (2) will be rewrit-

ten as:

$$F(x, z, z', \ldots, z^{(n-1)}) = 0, \tag{3}$$

i.e. it represents an equation of order $n - 1$ with respect to the function $z(x)$.

We must substitute any solution $z(x)$, $a < x < b$, of this equation into the differential equation $y' = z(x)$ and solve the latter with respect to y:

$$y = \int z(x)\, dx + C.$$

There has appeared an arbitrary constant. Some solutions of differential equation (3), not necessarily all the solutions, form a family of functions

$$z = \varphi(x, C_1, \ldots, C_{n-1}), \quad a < x < b,$$

depending on $n - 1$ parameters $C_1, \ldots, C_{n-1}$. There corresponds to it the family of solutions y of differential equation (2)

$$y = \int \varphi(x, C_1, \ldots, C_{n-1})\, dx + C_n,$$

depending on n parameters $C_1, \ldots, C_n$ ($C_n = C$).

Example 1. $y'' = \sqrt{1 + y'^2}$.

Here the function $y(x)$ does not enter explicitly into the equation. Setting $z = y'$, we find $z' = y''$ and our equation takes the form $z' = \sqrt{1 + z^2}$. Separating the variables, we have

$$\frac{dz}{\sqrt{1+z^2}} = dx, \quad x + C_1 = \int \frac{dz}{\sqrt{1+z^2}} = \ln\left(z + \sqrt{1+z^2}\right),$$

i.e.

$$z + \sqrt{1+z^2} = e^{x+C_1}, \quad 1 + z^2 = e^{2(x+C_1)} - 2ze^{x+C_1} + z^2,$$

$$z = \frac{1}{2}\left(e^{x+C_1} - e^{-(x+C_1)}\right) = \sinh(x + C_1).$$

But $z = y'$, hence, $dy = \sinh(x + C_1)\, dx$, $y = \cosh(x + C_1) + C_2$.

II. Let the left-hand member of equation (1) not involve explicitly the independent variable x:

$$F(y, y', \ldots, y^{(n)}) = 0. \tag{4}$$

We shall regard y in this equation as an independent variable, and y' as a sought-for function. Let us denote $y' = z(y)$.

Then,

$$y'' = \frac{dy'}{dx} = \frac{dz(y)}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = z'_y \cdot z,$$

$$y''' = \frac{dy''}{dx} = \frac{d(zz'_y)}{dx} = \frac{d(z \cdot z'_y)}{dy} \cdot \frac{dy}{dx} = z(z'^2_y + z \cdot z''_y),$$

. . . . . .

$$y^{(n)} = \omega(z, z'_y, \ldots, z^{(n-1)}_y).$$

Substituting these expressions into (4), we obtain a differential equation of order $n - 1$ with respect to z. Let $z = z(y)$, $\alpha < y < \beta$ be a solution of this differential equation different from zero on (α, β). Since $z(y) = \frac{dy}{dx}$, then

$$dx = \frac{dy}{z(y)}, \quad x = \int \frac{dy}{z(y)} + C, \quad \alpha < y < \beta.$$

We have obtained a solution $y(x)$ of the original equation (4) in an implicit form which depends on an arbitrary constant C.

But very often the functions $z = z(y)$ are obtained in the form of families of functions

$$z = z(y, C_1, \ldots, C_{n-1}), \quad \alpha < y < \beta,$$

dependent on $n - 1$ parameters $C_1, \ldots, C_{n-1}$. Solutions $y(x)$, corresponding to them, in their turn form a family of functions

$$y = y(x, C_1, \ldots, C_n)$$

dependent on n parameters $C_1, \ldots, C_n$ $(C_n = C)$.

Example 2. $y'^2 + 2yy'' = 0$.

Here x is not involved explicitly, therefore we set $y'_x = z(y)$, $y'' = \frac{dy'}{dx} = \frac{dz(y)}{dx} = z \cdot z'_y$. Substituting these expressions into the given equation, we have $z^2 + 2yzz'_y = 0$ or $z(z + 2yz'_y) = 0$. Hence $z = 0$ and $z + 2yz'_y = 0$.

If $z = 0$, then $\frac{dy}{dx} = 0$, $y = \text{const}$.

If $z + 2yz'_y = 0$, then separating the variables, we obtain

$$\frac{dz}{z} = -\frac{dy}{2y}, \quad \ln\left|\frac{z}{C_1}\right| = \ln\frac{1}{\sqrt{y}}, \quad z = \frac{C_1}{\sqrt{y}};$$

$$\frac{dy}{dx} = \frac{C_1}{\sqrt{y}}, \quad \sqrt{y}\,dy = C_1\,dx, \quad \frac{2}{3}y^{3/2} = C_1x + C_2.$$

III. The left-hand member of equation (1) is a homogeneous function of degree m with respect to the variables $y, y', \ldots, y^{(n)}$, i.e.

$$F(x, ty, ty', \ldots, ty^{(n)}) = t^m F(x, y, y', \ldots, y^{(n)}) \quad (t > 0).$$

To reduce the order, we introduce a new function $z(x)$ by the formula

$$y' = y \cdot z \qquad (y \neq 0).$$

Then

$$y'' = (yz)' = y'z + y \cdot z' = yz^2 + yz' = y(z^2 + z'),$$

. . . . . .

$$y^{(n)} = y\omega(z, z', \ldots, z^{(n-1)}).$$

Substituting these values into equation (1), we get

$$F(x, y, yz, y(z^2 + z'), \ldots, y\omega(z, z', \ldots, z^{(n-1)})) = 0$$

or, by the homogeneity of the function

$$y^m F(x, 1, z, z^2 + z', \ldots, \omega(z, z', \ldots, z^{(n-1)})) = 0.$$

Since $y \neq 0$, hence we obtain a differential equation of order $n - 1$

$$F(x, 1, z, \ldots, \omega(z, z', \ldots, z^{n-1})) = 0.$$

Let $z = z(x)$ be a solution of this equation. Since $z(x) = y'/y$,

$$\frac{dy}{y} = z(x)\,dx, \quad \ln\left|\frac{y}{C}\right| = \int z(x)\,dx,$$

$$y = C\exp\left(\int z(x)\,dz\right),$$

where C is an arbitrary constant. And if it turns out that

$$z = \varphi(x, C_1, \ldots, C_{n-1}),$$

then

$$y = C_n \exp\left\{\int \varphi(x, C_1, \ldots, C_{n-1})\, dx\right\},$$

where $C_1, \ldots, C_n$ are arbitrary constants.

Example 3. Solve the equation of the previous example using this method.

$F(x, y, y', y'') = y'^2 + 2yy''$ is a homogeneous function of second degree with respect to y, y', y''. The function $y(x) \equiv 0$ is a solution of the equation. We shall assume that $y \neq 0$. Setting $y' = z \cdot y$, we have $y'' = y(z' + z^2)$. Substituting these expressions into the equation, we get

$$(yz)^2 + 2y^2(z' + z^2) = 0 \qquad (y \neq 0).$$

Hence $3z^2 + 2z' = 0$. The function $z \equiv 0$ is a solution of the given equation (then $y' = 0$, $y = C$ is a solution of the original equation). Let $z \neq 0$, then

$$-\frac{2dz}{3z^2} = dx, \quad \frac{2}{3}z^{-1} = x + C_1, \quad z = \frac{2}{3}(x + C_1)^{-1};$$

$$\frac{dy}{y} = z\,dx = \frac{2}{3}\cdot\frac{dx}{x + C_1}, \quad \ln\left|\frac{y}{C_2}\right| = \frac{2}{3}\ln|x + C_1|,$$

$$y = C_2(x + C_1)^{2/3}$$

is the general solution. Note that the solution $y \equiv 0$ is obtained from the general solution for $C_2 = 0$.

Sec. 1.15. Linear Equations of Higher Order

A *linear differential equation of order n* is an equation of the form

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = f(x), \quad a < x < b, \qquad (1)$$

where $f(x)$, $p_0(x)$, $\ldots$, $p_{n-1}(x)$ are continuous functions defined on the interval (a, b). The left-hand member of equation (1) will be denoted by $L_n[y] \equiv L[y]$. It is called a *linear differential operator of order n*.

The operator $L(y)$ possesses the following properties:

(1) $L[Cy] = CL[y]$—homogeneity of the operator;

(2) $L[y_1 + y_2] = L[y_1] + L[y_2]$—additivity of the operator.

Notice that a homogeneous and additive operator is called *linear*.

On the basis of these properties, we readily obtain that

$$L\left[\sum_{k=1}^{m} C_k y_k\right] = \sum_{k=1}^{m} C_k L[y_k], \tag{2}$$

where C_k are arbitrary constants.

Example 1. Let $L[y] = y'' + y$.

It is easily seen that $L[\sin x] = -\sin x + \sin x = 0$, $L[x^2] = 2 + x^2$.

Equation (1) can be written in the form

$$L[y] \equiv L_n[y] = f(x), \quad a < x < b. \tag{1'}$$

If $f(x) \equiv 0$, then the equation

$$L_n[y] = 0, \quad a < x < b, \tag{3}$$

is called a *homogeneous linear differential equation of order n*. Then equation (1) is termed *nonhomogeneous*.

We suppose that the functions $f(x)$, $p_0(x)$, . . ., $p_{n-1}(x)$ are continuous on the interval (a, b). It is possible to prove that for such functions *differential equation* (1) *has a unique, defined on the same* (!) *interval* (a, b), *solution satisfying the initial conditions*

$$y(x_0) = y_0,\ y'(x_0) = y_0',\ \ldots,\ y^{(n-1)}(x_0) = y_0^{(n-1)}.$$

In this case, if the functions $p_j(x)$ *and* $f(x)$ *have continuous derivatives of order* q *on* (a, b), *then the indicated solution* $y(x)$ *has on* (a, b) *continuous derivatives up to orlder* $n + q$ *inclusively*.

Theorem 1. *If* $y_1, \ldots, y_m$ *are solutions of homogeneous equation* (3), *then their linear combination* $\sum_{k=1}^{m} C_k y_k$ *is also a solution of equation* (3).

This follows from equality (2).

Let us introduce the notion of linear dependence of functions by analogy with the corresponding notion for a system of vectors.

The functions $y_1(x), \ldots, y_m(x)$ are called *linearly dependent* on (a, b) if one of them is a linear combination of others $\forall x \in (a, b)$. In other words, the functions $y_1(x), \ldots$

. . ., $y_m(x)$ are called linearly dependent on (a, b) if there are numbers $\alpha_1, \ldots, \alpha_m$ at least one of which is not equal to zero such that

$$\alpha_1 y_1(x) + \ldots + \alpha_m y_m(x) \equiv 0, \quad a < x < b. \tag{4}$$

If identity (4) is fulfilled only in the case when all $\alpha_i = 0$, then the functions $y_1, \ldots, y_m$ are called *linearly independent* on (a, b).

A system of n linearly independent on an interval (a, b) solutions

$$y_1(x), \; y_2(x), \; \ldots, \; y_n(x)$$

of a homogeneous differential equation of order n (3) with coefficients $p_j(x)$, continuous on (a, b), is said to be a *fundamental system of solutions* of this equation.

To solve a homogeneous linear differential equation of order n (3) with continuous coefficients $p_j(x)$, it is necessary to find its fundamental system of solutions.

According to Theorem 1, an *arbitrary linear combination of solutions* $y_j(x)$, *i.e. the sum*

$$y = \sum_{k=1}^{n} C_k y_k(x), \tag{5}$$

where C_k *are arbitrary numbers, is again a solution of equation* (3) *on* (a, b). But it turns out that the converse is also true: any solution of differential equation (3) on an interval (a, b) is a linear combination of its particular solutions (independent of one another) $y_j(x)$ (see Theorem 4 below) forming a fundamental system of solutions.

Thus, the *general solution of homogeneous differential equation* (3) *has form* (5), *where* C_k *are arbitrary constants, and* $y_k(x)$ *are particular solutions of* (3) *forming a fundamental system of solutions of a homogeneous equation.*

Note that the general solution of a nonhomogeneous equation (1) is the sum of one of its particular solutions $y_0(x)$ and the general solution of the homogeneous equation

$$y(x) = y_0(x) + \sum_{j=1}^{n} C_j y_j(x). \tag{6}$$

Indeed,

$$L_n[y] = L_n[y_0] + \sum_{j=1}^{n} C_j L[y_j] = f(x) + 0 = f(x).$$

On the other hand, if y is an arbitrary solution of equation (1), then

$$L_n [y - y_0] = L_n [y] = L_n [y_0] = f(x) - f(x) = 0,$$

and, consequently, $y - y_0$ is a solution of a homogeneous equation; but then there are such numbers C_j that

$$y(x) - y_0(x) = \sum_{j=1}^{n} C_j y_j(x),$$

i.e. for these numbers equality (6) is fulfilled.

Theorem 2. *If the functions* $y_1(x), \ldots, y_m(x)$ *are linearly dependent on* (a, b) *and have derivatives up to order* $m - 1$, *then the determinant*

$$\begin{vmatrix} y_1(x) & \ldots & y_m(x) \\ y_1'(x) & \ldots & y_m'(x) \\ \ldots\ldots\ldots\ldots & & \\ y_1^{(m-1)}(x) & \ldots & y_m^{(m-1)}(x) \end{vmatrix} = 0 \quad (\forall x \in (a, b)). \tag{7}$$

Determinant (7) is called the *Wronski* determinant* or *Wronskian* and is denoted by the symbol $W(x) \equiv W[y_1, \ldots, y_m]$.

Proof. Since the functions $y_1(x), \ldots, y_m(x)$ are linearly dependent on (a, b), there exist such numbers (not all zero) $\alpha_1, \ldots, \alpha_m$ for which identity (4) is fulfilled on (a, b). Differentiating it $m - 1$ times, we obtain the system of equations

$$\begin{gathered} \alpha_1 y_1(x) + \ldots + \alpha_m y_m(x) \equiv 0, \\ \alpha_1 y_1'(x) + \ldots + \alpha_m y_m'(x) \equiv 0, \\ \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \\ \alpha_1 y_1^{(m-1)}(x) + \ldots + \alpha_m y_m^{(m-1)}(x) \equiv 0. \end{gathered}$$

By the hypothesis, this homogeneous system has a nontrivial solution $\alpha_1, \ldots, \alpha_m$ (that is, at least one $\alpha_i \neq 0$) for $\forall x \in (a, b)$. The latter is possible when the determinant of the system which is the Wronskian $W(x)$, is identically equal to zero.

The theorem has been proved.

* Wronski (also Hoene-Wronski), Joseph Maria (1778-1853). Polish mathematician (born in Poland, lived in France).

Remark 1. It follows from Theorem 2 that if $W(x) \not\equiv 0$ at least at one point of (a, b), then the functions $y_1, \ldots, y_m$ are linearly independent on (a, b).

Example 2. The functions $1, x, \ldots, x^{m-1}$ are linearly independent on any (a, b), since

$$W[1, x, \ldots, x^{m-1}] = \begin{vmatrix} 1 & x & x^2 & \ldots & x^{m-1} \\ 0 & 1! & 2x & \ldots & (m-1)\,x^{m-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & (m-1)! \end{vmatrix} = 1 \cdot 1!\ 2!\ \ldots\ (m-1)! \not\equiv 0.$$

Example 3. The functions $e^{k_1 x}, \ldots, e^{k_m x}$ are linearly independent on any (a, b) if $k_1, \ldots, k_m$ are distinct numbers.

Indeed,

$$W[e^{k_1 x}, \ldots, e^{k_m x}] = \begin{vmatrix} e^{k_1 x} & \ldots & e^{k_m x} \\ k_1 e^{k_1 x} & \ldots & k_m e^{k_m x} \\ \cdot & \cdot & \cdot \\ k_1^{m-1} e^{k_1 x} & \ldots & k_m^{m-1} e^{k_m x} \end{vmatrix}$$

$$= e^{(k_1 + \cdots + k_m)x} \begin{vmatrix} 1 & \ldots & 1 \\ k_1 & \ldots & k_m \\ \cdot & \cdot & \cdot \\ k_1^{m-1} & \ldots & k_m^{m-1} \end{vmatrix} \not\equiv 0,$$

since the last determinant is a Vandermonde determinant* which is not equal to zero for distinct $k_1, \ldots, k_m$.

Example 4. The functions $e^{kx}, xe^{kx}, \ldots, x^{m-1}e^{kx}$ are linearly independent on any (a, b).

Since $e^{kx} \not\equiv 0$ and

$$\alpha_1 e^{kx} + \alpha_2 x e^{kx} + \ldots + \alpha_m x^{m-1} e^{kx} = e^{kx}[\alpha_1 + \alpha_2 x + \ldots + \alpha_m x^{m-1}],$$

the linear independence of the indicated functions follows from Example 2.

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* See our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 2, Mir Publishers, Moscow, 1982.

Theorem 3. *For the solutions* $y_1(x), \ldots, y_n(x)$ *of a homogeneous linear differential equation* $L_n[y] = 0$ *with continuous coefficients to be linearly independent on* (a, b), *it is necessary and sufficient that* $W[y_1, \ldots, y_n] \neq 0$ *for all* $x \in (a, b)$.

Proof. (1) If $W(x) \neq 0$ on (a, b), then the functions $y_1(x), \ldots, y_n(x)$ are linearly independent irrespective of whether they are solutions of the equation $L_n[y] = 0$ (see Remark 1).

(2) Let $y_1, \ldots, y_n$ be linearly independent functions on (a, b) and let them be solutions of the equation $L_n[y] = 0$.

We are going to prove that $W(x) \neq 0$ everywhere on (a, b). Let us assume the contrary—that there is a point $x = x_0 \in [a, b]$, at which $W(x_0) = 0$. We then choose the points $\alpha_1, \ldots, \alpha_n$, not all zero, so that they are solutions of the system

$$\left.\begin{aligned} \alpha_1 y_1(x_0) + \ldots + \alpha_n y_n(x_0) &= 0, \\ \alpha_1 y_1'(x_0) + \ldots + \alpha_n y_n'(x_0) &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \alpha_1 y_1^{(n-1)}(x_0) + \ldots + \alpha_n y_n^{(n-1)}(x_0) &= 0. \end{aligned}\right\} \qquad (8)$$

This can be done, since the determinant of system (8) is $W(x_0) = 0$. Then, by virtue of Theorem 1, the function $y = \sum_{i=1}^{n} \alpha_i y_i(x)$ will be a solution of the equation $L_n[y] = 0$ with zero initial conditions

$$y(x_0) = 0,\ y'(x_0) = 0,\ \ldots,\ y^{(n-1)}(x_0) = 0 \qquad \text{(by (8))}.$$

But the same conditions are also satisfied by the trivial solution $y \equiv 0$. By virtue of the existence and uniqueness theorem, there can be only one solution satisfying these n initial conditions, consequently,

$\sum_{i=1}^{n} \alpha_i y_i(x) \equiv 0$ on (a, b), i.e. the functions $y_1, \ldots, y_n$ are linearly dependent on (a, b) which was not assumed.

The theorem has been proved.

If $p_i(x)$ are discontinuous functions in the interval where we seek for a solution, then the equation $L_n[y] = 0$ can have more than one solution satisfying the initial

conditions $y(x_0) = y'(x_0) = \ldots = y^{(n-1)}(x_0) = 0$ and then it is possible that $W(x) \equiv 0$ on (a, b).

Example 5. It is readily verified that the functions

$$y_1(x) = \begin{cases} (x-1)^\alpha, & 0 < x \leqslant 1, \\ 0, & 1 < x < 2, \end{cases}$$

$$y_2(x) = \begin{cases} 0, & 0 < x \leqslant 1, \\ (x-1)^\alpha, & 1 < x < 2 \end{cases} \quad (\alpha > 2)$$

are linearly independent on (0, 2) and for them $W(x) \equiv 0$ on (0, 2).

This is due to the fact that the function $y = C_1 y_1 + C_2 y_2$ is the general solution of the equation

$$y'' - \frac{\alpha(\alpha-1)}{(x-1)^2} y = 0,$$

where $p_0(x) = -\frac{\alpha(\alpha-1)}{(x-1)^2}$ is discontinuous at the point $x = 1$. For this equation the existence and uniqueness theorem does not hold (in the neighbourhood of the point $x = 1$). Not only the function $y \equiv 0$, but also the function $y_1(x)$ are solutions of the differential equation satisfying the conditions $y = 0$ and $y' = 0$ for $x = 1$.

Theorem 4 (structure of a general solution). *If $y_1, \ldots, y_n$ are linearly independent on (a, b) solutions of a homogeneous linear differential equation of order n $L_n[y] = 0$ with continuous coefficients $p_i(x)$, $a < x < b$, then the function*

$$y(x) = \sum_{k=1}^{n} C_k y_k(x), \quad a < x < b, \tag{9}$$

where C_k are arbitrary constants, is the general solution of the equation $L_n[y] = 0$, i.e. sum (9) is a solution of this equation for any C_k, and, conversely, any solution of this equation is representable in the form of sum (9) for appropriate values of C_k.

Proof. We already know that sum (9) is a solution of the equation $L_n[y] = 0$ for any C_k. Let, conversely, $z = z(x)$ be an arbitrary solution of this equation. We set

$$z(x_0) = z_0, \quad z'(x_0) = z_0', \quad \ldots, \quad z^{(n-1)}(x_0) = z_0^{(n-1)}. \tag{10}$$

For the obtained numbers $z_0, z_0', \ldots, z_0^{(n-1)}$ we form a linear system of equations with respect to the unknown

numbers C_k:

$$\left.\begin{aligned} &\sum_{k=1}^{n} C_k y_k(x_0) = z_0, \\ &\cdots\cdots\cdots\cdots \\ &\sum_{k=1}^{n} C_k y_k^{(n-1)}(x_0) = z_0^{(n-1)}. \end{aligned}\right\} \tag{11}$$

The determinant of system (11) $W(x_0)$ is not equal to zero, since the functions $y_1, \ldots, y_n$ are solutions of the equation $L_n[y] = 0$ linearly independent on (a, b). Therefore, there is a unique system of numbers $C_1^0, \ldots, C_n^0$ satisfying equations (11). Substituting them into (9), we obtain a solution of our equation in the form

$$y = \sum_{k=1}^{n} C_k^0 y_k(x),$$

satisfying the same initial conditions (10) satisfied by $z(x)$. But then, by virtue of the existence and uniqueness theorem, there takes place the equality $z(x) = y(x)$, $x \in (a, b)$.

The theorem has been proved.

Thus, in order to find the general solution of a homogeneous equation $L_n[y] = 0$, it suffices to find some n linearly independent solutions of this equation, and then the general solution will be their linear combination (9). We recall here that any set of n linearly independent particular solutions of the equation $L_n[y] = 0$ is conventionally called a fundamental system of solutions of this equation.

There arises a question whether a fundamental system of solutions always exists for a given homogeneous differential equation (3) with continuous coefficients. Let us show that it does exist.

We take n vectors

$$\begin{aligned} \mathbf{a}^{(1)} &= (1, 0, \ldots, 0), \\ \mathbf{a}^{(2)} &= (0, 1, 0, \ldots, 0), \\ &\cdots\cdots\cdots\cdots \\ \mathbf{a}^{(n)} &= (0, 0, \ldots, 0, 1) \end{aligned}$$

and associate each of them with a solution $y_k(x)$ of equation (3). Namely, let $y_k(x)$ be a solution satisfying the

following initial conditions:

$$y_k(x_0) = 0,\ y_k'(x_0) = 0,\ \ldots,\ y_k^{(k-2)}(x_0) = 0,\ y_k^{(k-1)}(x_0) = 1,$$
$$y_k^{(k)}(x_0) = 0,\ \ldots,\ y_k^{(n-1)}(x_0) = 0 \quad (k = 1, 2, \ldots, n).$$

The Wronskian $W(x)$ for this system of solutions for $x = x_0$ $(a \leqslant x_0 \leqslant b)$, is, obviously, the determinant of the matrix formed from the vectors $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}$. It is equal to 1:

$$W(x_0) = 1 \neq 0.$$

But then the system of solutions $y_1, \ldots, y_n$ is linearly independent, since for a dependent system the Wronskian would be identically equal to zero.

Sec. 1.16. Homogeneous Linear Equations of *n*th Order with Constant Coefficients

The equation

$$L_n[y] = y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_1y' + p_0y = 0, \quad (1)$$
$$-\infty < x < \infty,$$

where p_i $(i = 0, 1, \ldots, n-1)$ are constant numbers, is called a *homogeneous linear differential equation of nth order with constant coefficients.*

To solve equation (1), it is necessary to find one of its fundamental systems of solutions, i.e. to find some n of its particular solutions $y_1(x), \ldots, y_n(x)$ forming a linearly independent system throughout the entire axis, and then (see the preceding section) the general solution of equation (1) will have the form

$$y = C_1y_1(x) + \ldots + C_ny_n(x),$$

where $C_1, \ldots, C_n$ are arbitrary constant numbers.

We shall look for particular solutions of equation (1) in the form $y = e^{kx}$, where k is a constant number. Then

$$y' = ke^{kx},\quad y'' = k^2e^{kx},\ \ldots,\ y^{(n)} = k^{(n)}e^{kx}.$$

Substituting these expressions into (1), we obtain

$$L_n[e^{kx}] = e^{kx}[k^n + p_{n-1}k^{n-1} + \ldots + p_1k + p_0] = 0.$$

Since $e^{kx} \neq 0$, we have

$$R_n(k) \equiv k^n + p_{n-1}k^{n-1} + \ldots + p_1 k + p_0 = 0. \quad (2)$$

Thus, if k is a root of algebraic equation (2), then the function $y = e^{kx}$ is a solution of equation (1), and conversely. Equation (2) is called *characteristic* for equation (1). This equation can be obtained from (1) by replacing the derivatives $y^{(j)}$ by the respective numbers k^j ($j = 0, 1, \ldots, n$). Conversely, equation (1) is readily restored by equation (2). As is known, equation (2) has n roots taking into account multiplicity.

1°. Let the roots $k_1, \ldots, k_n$ of equation (2) be distinct. Then n functions

$$e^{k_1 x}, \; e^{k_2 x}, \; \ldots, \; e^{k_n x} \quad (3)$$

are solutions of equation (1), linearly independent on $(-\infty, \infty)$ (see Example 3, Sec. 1.15), i.e. they form a fundamental system of solutions of equation (1). Therefore, in this case (see Theorem 4 proved in the preceding section) the general solution of equation (1) will be written in the form

$$y(x) = \sum_{j=1}^{n} C_j e^{k_j x}, \quad (4)$$

where C_j are arbitrary constants.

Example 1. The equation $y'' - 5y' + 6y = 0$ has the characteristic equation $k^2 - 5k + 6 = 0$ with simple real roots $k_1 = 2$, $k_2 = 3$. The general solution is $y = C_1 e^{2x} + C_2 e^{3x}$.

Example 2. The equation $y''' - 2y'' - y' + 2y = 0$ has the characteristic equation $k^3 - 2k^2 - k + 2 = 0$ or $(k - 2)(k^2 - 1) = 0$. The roots are simple: $k_1 = 2$, $k_2 = 1$, $k_3 = -1$. To them there corresponds the general solution of the given equation $y = C_1 e^{2x} + C_2 e^{x} + C_3 e^{-x}$.

If the numbers p_l are real and if some root k_j is a complex one ($k_j = \alpha + i\beta$, $\beta \neq 0$), then among the remaining roots there must be its conjugate root ($k_s = \alpha - i\beta$, $s \neq j$). Since the complex functions $e^{(\alpha+i\beta)x}$, $e^{(\alpha-i\beta)x}$ are solutions of equation (1), then (see Theorem 1, Sec. 1.15) the function

$$\frac{1}{2}\left[e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}\right] = e^{\alpha x}\cos\beta x, \quad (5)$$

as well as the function

$$\frac{1}{2i}\left[e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}\right] = e^{\alpha x}\sin\beta x, \tag{6}$$

are, in their turn, solutions of equation (1). We have used here Euler's formulas $e^{\pm ix} = \cos x \pm i \sin x$.

Solutions (5) and (6) are real functions and in this they have an advantage over the functions $\exp(k_j x)$, $\exp(k_s x)$, therefore the latter functions are often replaced by them in system (3).

It is possible to prove that modified system (3) is linearly independent on $(-\infty, \infty)$.

Example 3. To the equation $y'' + y = 0$ there corresponds the characteristic equation $k^2 + 1 = 0$ and conjugate complex roots $k_1 = i$, $k_2 = -i$. In the exponential form the solution is written as

$$y = C_1 e^{ix} + C_2 e^{-ix}.$$

Using Euler's formulas, we can write the general solution as follows

$$\begin{aligned} y &= C_1 (\cos x + i \sin x) + C_2 (\cos x - i \sin x) \\ &= (C_1 + C_2)\cos x + i (C_1 - C_2)\sin x \\ &= A \cos x + B \sin x. \end{aligned}$$

Here A and B are arbitrary constants, since the equations $A = C_1 + C_2$, $B = i(C_1 - C_2)$ can be solved with respect to C_1, C_2 for any A, B.

Example 4. To the equation $y'' + y' + y = 0$ there corresponds the characteristic equation $k^2 + k + 1 = 0$ with the roots $k_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

The general solution of the given equation will be written as:

$$y = e^{-x/2}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right).$$

2°. If k_1 is a root of multiplicity m, then in system (3) there appears m equal functions

$$e^{k_1 x},\ e^{k_2 x},\ \ldots,\ e^{k_m x} \quad (k_1 = k_2 = \ldots = k_m) \tag{7}$$

and system (3) ceases to be linearly independent. It turns out (see Lemma 1 below) that in this case (when k_1 is a multiple root) the functions

$$e^{k_1 x},\ xe^{k_1 x},\ \ldots,\ x^{m-1}e^{k_1 x} \tag{8}$$

are also solutions of equation (1) forming a system linearly independent on any interval (a, b) (see Lemma 1 below). They usually replace functions (7) in system (3).

If all the multiple roots are replaced in such a way, then the newly obtained system will be, as it can be proved, linearly independent.

Example 5. To the equation $y'' - 2y' + y = 0$ there corresponds the characteristic equation $k^2 - 2k + 1 = 0$ having one double root $k = 1$. Therefore the general solution of this equation is written in the form $y = C_1 e^x + C_2 x e^x$.

Example 6. The equation $y''' + 3y'' + 3y' + y = 0$ has the characteristic equation $k^3 + 3k^2 + 3k + 1 = 0$ or $(k + 1)^3 = 0$, whence $k = -1$ which is a triple root and, consequently, the general solution has the form $y = e^x (C_1 + C_2 x + C_3 x^2)$.

For real p_l, if $k_1 = \alpha + i\beta$ is a complex root of the order of multiplicity m, then $\bar{k}_1 = \alpha - i\beta$ is also a root of multiplicity m to which there correspond the solutions

$$e^{\bar{k}_1 x},\ xe^{\bar{k}_1 x},\ \ldots,\ x^{m-1}e^{\bar{k}_1 x}, \tag{8'}$$

appearing in series (3). Let us write solutions (8) and (8') in a more detailed form

$$e^{(\alpha+i\beta)x},\ xe^{(\alpha+i\beta)x},\ \ldots,\ x^{m-1}e^{(\alpha+i\beta)x},$$
$$e^{(\alpha-i\beta)x},\ xe^{(\alpha-i\beta)x},\ \ldots,\ x^{m-1}e^{(\alpha-i\beta)x}.$$

Adding them and dividing by 2 or subtracting and dividing by $2i$, we obtain two systems of real solutions of equation (1)

$$\left.\begin{aligned} &e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ \ldots,\ x^{m-1}e^{\alpha x}\cos\beta x,\\ &e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ \ldots,\ x^{m-1}e^{\alpha x}\sin\beta x, \end{aligned}\right\} \tag{9}$$

by which complex functions (8) and (8') are replaced in series (3).

Note that it is possible to prove that if a similar change is carried out for all multiple roots of the characteristic

equation, then the functions of the new system form a linearly independent system of solutions of equation (1) on any interval (a, b).

Example 7. The differential equation $y^{(4)} + 2y'' + y = 0$ has the characteristic equation $k^4 + 2k^2 + 1 = 0$ or $(k^2 + 1)^2 = 0$, whence $k_1 = i$ and $k_2 = -i$ which are roots of the second multiplicity. Therefore the general solution of the given differential equation has the form

$$y = C_1 \cos x + C_2 x \cos x + C_3 \sin x + C_4 x \sin x.$$

Lemma 1. *If k_1 is a root of the characteristic equation $R_n(k) = 0$ of multiplicity m, then the functions*

$$e^{k_1 x}, \; xe^{k_1 x}, \; \ldots, \; x^{m-1} e^{k_1 x}$$

are solutions of differential equation (1), *linearly independent on any interval* (a, b).

Proof. Linear independence of the indicated functions was established in Example 4 of the preceding section.

Since k_1 is a root of multiplicity m of the polynomial $R_n(k)$, $R_n(k) = (k - k_1)^m \varphi(k)$, where $\varphi(k)$ is a polynomial such that $\varphi(k_1) \neq 0$. Hence,

$$R_n(k_1) = 0, \quad R_n'(k_1) = 0, \; \ldots, \; R_n^{(m-1)}(k_1) = 0. \quad (10)$$

We have

$$L_n[x^j e^{kx}] = L_n\left[\frac{\partial^j}{\partial k^j} e^{kx}\right] = \frac{\partial^j}{\partial k^j} L_n[e^{kx}]$$

$$= \frac{\partial^j}{\partial k^j}[e^{kx} R_n(k)] = \sum_{s=0}^{j} C_s \frac{\partial^{j-s} e^{kx}}{\partial k^{j-s}} \cdot \frac{d^s R_n(k)}{dk^s},$$

where C_s are constant numbers. Hence, if $j \leqslant m - 1$, and $k = k_1$, then, by virtue of (10), we obtain $L_n[x^j e^{k_1 x}] = 0$.

The lemma has been proved.

Remark 1. It follows from the aforesaid that if k_j is a root of the characteristic equation of multiplicity l_j $(j = 1, \ldots, m; \sum_{j=1}^{m} l_j = n)$, then fundamental system of solutions of equation (1) consists of the functions

$$e^{k_j x}, \; xe^{k_j x}, \; \ldots, \; x^{l_j - 1} e^{k_j x} \quad (j = 1, \ldots, m).$$

If some root k_j is a complex one: $k_j = \alpha_j + i\beta_j$ ($\beta_j \neq 0$), then to this root, together with the root $\bar{k}_j$, there corresponds the group of real functions

$$e^{\alpha_j x}\cos\beta_j x,\ \ldots,\ x^{l_j-1}e^{\alpha_j x}\cos\beta_j x,$$
$$e^{\alpha_j x}\sin\beta_j x,\ \ldots,\ x^{l_j-1}e^{\alpha_j x}\sin\beta_j x.$$

Example 8. An equation with variable coefficients of the form

$$x^n y^{(n)} + p_{n-1}x^{n-1}y^{(n-1)} + \ldots + p_1 xy' + p_0 y = 0,$$

where $p_0, p_1, \ldots, p_{n-1}$ are constant numbers, is called *Euler's equation.*

With the aid of the substitution $x = e^t$ this equation is reduced to an equation with constant coefficients. Indeed, we have

$$y(x) = y(e^t) = v(t).$$

Hence,

$$y'(x) = v'(t)\cdot\frac{dt}{dx} = v'(t)\cdot e^{-t},\quad xy'(x) = v'(t);$$

$$y''(x) = v''(t)\,e^{-2t} - v'(t)\,e^{-2t};\ x^2y''(x) = v''(t) - v'(t);\ldots.$$

Substituting these expressions, we obtain an equation with constant coefficients with respect to the function $v(t)$. The particular solutions of this equation, as it was shown above, are functions of the form $e^{kt} = x^k$ or $t^j e^{kt} = x^k \ln^j x$, where k is the root (simple or multiple) of the corresponding characteristic equation. Thus, particular solutions of Euler's equation can be looked for at once in the form x^k, $x^k \ln^j x$.

Example 9. Solve a concrete Euler's equation: $x^2y'' + 3xy' + y = 0$. We shall look for particular solutions in the form $y = x^k$, then

$$y' = kx^{k-1},\quad y'' = k(k-1)x^{k-2}.$$

Substituting these expressions for the derivatives, we get

$$x^2k(k-1)x^{k-2} + 3kx\cdot x^{k-1} + x^k$$
$$= x^k[k(k-1) + 3k + 1] = 0.$$

Hence, if $x \neq 0$, then $k(k-1) + 3k + 1 = 0$. The last equation has the root $k = -1$ (of second multiplicity).

Hence, $y = 1/x$ is a solution of Euler's equation. Another solution is $y = (\ln x)/x$ which is made sure directly. Since $1/x$ and $(\ln x)/x$ are linearly independent (their Wronskian is equal to $1/x^3 \neq 0$), then

$$y = \frac{C_1}{x} + C_2 \frac{\ln x}{x}$$

is the general solution of the given Euler equation.

Sec. 1.17. Method of Variation of Constants (Parameters)

Consider a nonhomogeneous linear equation of nth order

$$L_n [y] = y^{(n)} + p_{n-1} y^{(n-1)} + \ldots + p_0 y = f(x), \quad (1)$$

where the coefficients $p_i = p_i(x)$ and the right-hand member $f(x)$ are continuous functions defined on the interval (a, b).

Suppose we know the fundamental system of solutions $y_1(x), \ldots, y_n(x)$ of the corresponding homogeneous equation

$$L_n [y] = 0. \quad (2)$$

As it was shown in Sec. 1.15 (formula (6)), the general solution of equation (1) is equal to the sum of the general solution of equation (2) and a certain solution of equation (1).

The solution of nonhomogeneous equation (1) can be obtained using the *method of variation of constants*, provided the general solution of homogeneous equation (2) is known. We are going to explain this method taking as an example a third-order equation.

And so, let there be given a third-order linear equation

$$y''' + p_2 y'' + p_1 y' + p_0 y = f(x). \quad (3)$$

Let the general solution of the corresponding homogeneous equation be

$$y = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x), \quad (4)$$

where y_1, y_2, y_3 are linearly independent solutions of equation (2) ($L [y_i] = 0$, $i = 1, 2, 3$).

We shall look for a solution of nonhomogeneous equation (3) in the form of sum (4), where $C_1(x)$, $C_2(x)$, $C_3(x)$ are some continuously differentiable functions to be found. Let us impose on the sought-for functions $C_i(x)$ the following two conditions:

$$\left.\begin{aligned} \sum_{i=1}^{3} C_i'(x)\, y_i(x) = 0, \\ \sum_{i=1}^{3} C_i'(x)\, y_i'(x) = 0. \end{aligned}\right\} \tag{5}$$

We then obtain

$$y' = \sum_{i=1}^{3} C_i y_i' + \sum_{i=1}^{3} C_i' y_i = \sum_{i=1}^{3} C_i y_i',$$

$$y'' = \sum_{i=1}^{3} C_i y_i'' + \sum_{i=1}^{3} C_i' y_i' = \sum_{i=1}^{3} C_i y_i'',$$

$$y''' = \sum_{i=1}^{3} C_i y_i''' + \sum_{i=1}^{3} C_i' y_j''.$$

Substituting these derivatives and the function y itself into (3), we get

$$\sum_{i=1}^{3} C_i y_i''' + \sum_{i=1}^{3} C_i' y_i'' + p_2 \sum_{i=1}^{3} C_i y_i'' + p_1 \sum_{i=1}^{3} C_i y_i' + p_0 \sum_{i=1}^{3} C_i y_i = f(x),$$

or

$$C_1 (y_1''' + p_2 y_1'' + p_1 y_1' + p_0 y_1) + C_2 (y_2''' + p_2 y_2'' + p_1 y_2' + p_0 y_2) + C_3 (y_3''' + p_2 y_3'' + p_1 y_3' + p_0 y_3) + \sum_{i=1}^{3} C_1' y_i'' = f(x).$$

But the expressions in the parentheses standing in the left-hand side of this equality are equal to zero, therefore

$$\sum_{i=1}^{3} C_i' y_i'' = f(x). \tag{6}$$

We have obtained equation (6) and two equations (5) with the coefficients y_i, y_i', y_i'' and right-hand member $f(x)$ which are continuous on (a, b). These three equations form a linear algebraic system with respect to the unknowns C_1', C_2', C_3' with a nonzero determinant since this is the Wronskian for a fundamental system of solutions y_1, y_2, y_3. Therefore the given system has a unique solution

$$C_i'(x) = \varphi_i(x) \quad (i = 1, 2, 3),$$

where φ_i are functions continuous on (a, b). Whence

$$C_i(x) = \int \varphi_i(x)\, dx, \tag{7}$$

the functions $C_i(x)$ having a continuous derivative on (a, b). Consequently, a particular solution of nonhomogeneous equation (1) has the form

$$y(x) = C_1(x)\, y_1(x) + C_2(x)\, y_2(x) + C_3(x)\, y_3(x),$$

where the functions $C_i(x)$ are defined by equalities (7).

Example. $y'' - 3y' + 2y = e^{3x}$, $R_2(k) = k^2 - 3k + 2 = 0$, $k_1 = 1$, $k_2 = 2$ are the roots of the characteristic equation, $y = C_1 e^x + C_2 e^{2x}$ being the general solution of the homogeneous equation.

Let us find the particular solution of the nonhomogeneous equation, applying the method of variation of the constants C_1 and C_2. Let us form system (5), (6):

$$\left.\begin{aligned} C_1'(x)\, e^x + C_2'(x)\, e^{2x} &= 0, \\ C_1'(x)\, e^x + C_2'(x)\, 2 \cdot e^{2x} &= e^{3x}, \end{aligned}\right\} W[y_1, y_2] = e^{3x} \neq 0.$$

Solving this system, we have $C_1'(x) = -e^{2x}$, $C_2'(x) = e^x$ Hence, $C_1(x) = -e^{2x}/2$, $C_2(x) = e^x$ and the particular solution

$$\bar{y} = -\frac{1}{2} e^{2x} \cdot e^x + e^x e^{2x} = \frac{1}{2} e^{3x}.$$

Thus, the general solution of the above equation has the form

$$y = C_1 e^x + C_2 e^{2x} + \frac{1}{2} e^{3x}.$$

Sec. 1.18. Particular Solution of a Nonhomogeneous Differential Equation with Constant Coefficients

Consider a nonhomogeneous equation of nth order with constant coefficients

$$y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_0 y = f(x), \quad -\infty < x < \infty, \tag{1}$$

where the right-hand member has a special form $f(x) = e^{k_0 x}$ or a more general form $f(x) = x^{l-1}e^{k_0 x}$, or also $x^{l-1}e^{\alpha x}\cos \beta x$, or $x^{l-1}e^{\alpha x}\sin \beta x$ $(l = 1, 2, \ldots)$.

We shall be interested in the methods of finding the particular solution of equation (1). This is of importance, since, to obtain the general solution of equation (1), we have to find one of its particular solutions and add to it the general solution of the corresponding homogeneous equation, which we already know how to find (see Sec. 1.15).

Let us first consider the simplest case $f(x) = ae^{k_0 x}$. The characteristic polynomial of equation (1) has the form

$$R_n(k) = k^n + p_{n-1}k^{n-1} + \ldots + p_0.$$

If k_0 is not a root of the characteristic equation

$$R_n(k) = 0, \tag{2}$$

then the particular solution of the equation

$$L_n[y] = y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_0 y = ae^{k_0 x} \tag{3}$$

can be found in the form of the function

$$y = Ae^{k_0 x}, \tag{4}$$

where A is a constant. Substituting this function into equation (3), we get $AR_n(k_0)\,e^{k_0 x} = ae^{k_0 x}$, whence, after reduction by a nonzero factor $e^{k_0 x}$, we obtain $AR_n(k_0) = a$ or

$$A = a/R_n(k_0), \tag{5}$$

and the number A has been found.

Example 1. The equation

$$y'' + y = e^x \tag{6}$$

has the characteristic equation

$$R_2(k) = k^2 + 1 = 0.$$

The number $k_0 = 1$ is not a root of the characteristic equation ($R_2(k_0) = R_2(1) = 2 \neq 0$), therefore particular solution (6) may be sought in the form $y = Ae^x$. According to (5), $A = (R_2(1))^{-1} = 1/2$. The general solution of equation (6) has the form

$$y = \frac{1}{2} e^x + C_1 \cos x + C_2 \sin x,$$

where C_1, C_2 are arbitrary constants.

If k_0 is a root of characteristic equation (2), then equation (3), obviously, has no solution of form (4). In this case the following lemma will help us.

Lemma 1. *If k_0 is a real or complex root of multiplicity m of characteristic equation* (2), *then the particular solution of differential equation* (3) *can be found in the form*

$$y = Ax^m e^{k_0 x},$$

where A is a constant.

Let us first give an example.

Example 2.

$$y'' - 2y' + y = e^x. \tag{7}$$

The characteristic equation is $k^2 - 2k + 1 = 0$, or $(k-1)^2 = 0$. The number $k_0 = 1$ is its root of multiplicity 2. Therefore, according to Lemma 1, the solution should be sought in the form

$$y = Ax^2 e^x.$$

We have

$$y' = A \cdot 2xe^x + Ax^2 e^x,$$

$$y'' = 2Ae^x + 4Axe^x + Ax^2 e^x.$$

Substituting into (7), we obtain

$$2Ae^x = e^x.$$

Consequently, $A = 1/2$ and the general solution is

$$y = \frac{1}{2} x^2 e^x + (C_1 + C_2 x) e^x.$$

Proof of Lemma 1. Let us apply the operation (operator) to both members of differential equation (3):

$$\frac{d}{dx} - k_0 \left(\left(\frac{d}{dx} - k_0 \right) u = \frac{du}{dx} - k_0 u \right).$$

Then we shall have in the right-hand side:

$$\left(\frac{d}{dx} - k_0 \right) a e^{k_0 x} = \frac{d}{dx} (a e^{k_0 x}) - k_0 a e^{k_0 x} = 0.$$

Therefore

$$\left(\frac{d}{dx} - k_0 \right) L_n [y] = 0. \tag{8}$$

We have obtained a differential equation of order $n + 1$ with constant coefficients. Its characteristic equation has the form

$$(k - k_0) R_n (k) = 0. \tag{9}$$

The equations $R_n (k) = 0$ and (9) have the same roots, but the root k_0 in equation (9) is of multiplicity $m + 1$ (exceeds by unity). The general solution of the homogeneous equation

$$L_n [y] = 0 \tag{10}$$

will be written as follows:

$$v = (C_1 + C_2 x + \ldots + C_m x^{m-1}) e^{k_0 x} + \sum_{j=m+1}^{n} C_j y_j (x),$$

where $y_{m+1}, \ldots, y_n$ are solutions corresponding to the roots of the characteristic equation not equal to k_0 (if any). And the general solution of equation (8) can be written in the form

$$z = A x^m e^{k_0 x} + v.$$

Any solution y of equation (3) is a solution of equation (8), and therefore

$$y = A x^m e^{k_0 x} + v \tag{11}$$

for a certain set of constants

$$C_1, C_2, \ldots, C_n, A. \tag{12}$$

We have shown that if y is a solution of equation (3) $L_n[y] = ae^{k_0 x}$, then constants (12) can be chosen so that $y = Ax^m e^{k_0 x} + v$. Now, for the indicated constants (12), we shall have

$$L_n[Ax^m e^{k_0 x}] = L_n[y - v] = L_n[y] - L_n[v]$$
$$= ae^{k_0 x} - 0 = ae^{k_0 x}.$$

This proves that there is a number A such for which the function

$$y = Ax^m e^{k_0 x}$$

is a solution of nonhomogeneous differential equation (3).

The lemma has been proved.

Let us now consider two differential equations

$$y_1^{(n)} + p_{n-1}y_1^{(n-1)} + \ldots + p_0 y_1 = ax^{l-1}e^{\alpha x}\cos\beta x, \qquad (13)$$

$$y_2^{(n)} + p_{n-1}y_2^{(n-1)} + \ldots + p_0 y_2 = ax^{l-1}e^{\alpha x}\sin\beta x, \qquad (14)$$

where α, β, p_j are real numbers, and l is natural. Multiplying the second equation by i and adding it to the first one, we obtain the equation

$$z^{(n)} + p_{n-1}z^{(n-1)} + \ldots + p_0 z = ax^{l-1}e^{k_0 x}, \qquad (15)$$

$$z = y_1 + iy_2, \quad k_0 = \alpha + i\beta.$$

Instead of two equations (13) and (14) with unknown functions $y_1(x)$, $y_2(x)$, we have obtained one equation (15) with an unknown complex function $z(x) = y_1(x) + iy_2(x)$. If the solution $z = y_1 + iy_2$ of equation (15) is found, then its real part y_1 will be a solution of equation (13), and the imaginary part y_2 of equation (14).

For $l = 1$ equation (15) was already investigated.

Example 3. Given the equation

$$y'' + y = \sin x. \qquad (16)$$

Along with it, let us consider the equation

$$y'' + y = \cos x. \qquad (17)$$

We have

$$y_1'' + y_1 + \cos x, \quad y_2'' + y_2 = \sin x.$$

Multiplying the second equation by i and then adding it to the first, we obtain

$$z'' + z = e^{ix}, \quad z = y_1 + iy_2. \tag{18}$$

The number $k_0 = i$ is a root of multiplicity 1 of the characteristic equation, therefore the particular solution should be sought for in the form $z = Axe^{ix}$. We have

$$z' = Ae^{ix} + Aixe^{ix}, \quad z'' = 2Aie^{ix} - Axe^{ix}.$$

Substituting z, z', z'' into equation (18), we get

$$2Aie^{ix} = e^{ix}, \quad A = 1/(2i) = -i/2.$$

The particular solution of equation (18) has the form

$$z = -\frac{ix}{2} e^{ix} = -\frac{ix}{2} (\cos x + i \sin x) = \frac{x}{2} \sin x - i \frac{x}{2} \cos x.$$

Its imaginary part

$$y = -\frac{x}{2} \cos x$$

is the particular solution of the given equation (16), while the real part is the particular solution of equation (17).

The case $l > 1$ (for equation (15)) is provided in the following lemma.

Lemma 2. *The particular solution of equation* (15) *can be found among the functions of the form*

$$(C_m x^m + \ldots + C_{m+l-1} x^{m+l-1}) e^{k_0 x}, \tag{19}$$

where m is multiplicity of the root k_0 of the characteristic equation $R_n(k) = 0$, and C_m, C_{m+1}, . . ., C_{m+l-1} are constants.

If k_0 is not a root of the equation $R_n(k) = 0$, then, formally, we may still regard it as a root, but of multiplicity zero. In this case we have to put $m = 0$ in formula (19) and look for the particular solution in the form

$$(C_0 + C_1 x + \ldots + C_{l-1} x^{l-1}) e^{k_0 x}.$$

This lemma is proved much in the same manner as Lemma 1.

The general solution of the homogeneous equation $L_n[z] = 0$ is written in the following way:

$$v = (C_0 + C_1 x + \ldots + C_{m-1}x^{m-1})\, e^{k_0 x} + \sum_{j=m+1}^{n} C_j z_j(x), \quad (20)$$

where $z_{m+1}, \ldots, z_n$ are solutions corresponding to the roots not equal to k_0, if any.

Let us subject equation (15) to the operation $\left(\frac{d}{dx} - k_0\right)^l$. Since

$$\left(\frac{d}{dx} - k_0\right)(ax^{l-1}e^{k_0 x}) = \frac{d}{dx}(ax^{l-1}e^{k_0 x}) - k_0 a x^{l-1}e^{k_0 x}$$

$$= a(l-1)\, x^{l-2}e^{k_0 x},$$

we have

$$\left(\frac{d}{dx} - k_0\right)^l (ax^{l-1}e^{k_0 x}) = \left(\frac{d}{dx} - k_0\right)(a(l-1)!\, e^{k_0 x}) = 0.$$

Therefore we obtain a homogeneous differential equation with constant coefficients

$$\left(\frac{d}{dx} - k_0\right)^l L_n[z] = 0. \quad (21)$$

Its characteristic equation has the same roots as the equation $R_n(k) = 0$, the multiplicity of k_0 exceeding that of the common roots by the number l. Therefore the general solution of equation (21) can be written in the form

$$z = (C_0 + C_1 x + \ldots + C_{m+l-1}x^{m+l-1})\, e^{k_0 x} + \sum_{h=m+1}^{n} C_h z_h$$

$$= v + (C_m x^m + \ldots + C_{m+l-1}x^{m+l-1})\, e^{k_0 x}. \quad (22)$$

Any solution z of equation (15) is a solution of equation (21), and therefore it can be written in form (22). But then

$$L_n[(C_m x^m + \ldots + C_{m+l-1}x^{m+l-1})\, e^{k_0 x}]$$

$$= L_n[z - v] = L_n[z] - L_n[v] = ax^{l-1}e^{k_0 x},$$

which shows that function (19) is a particular solution of equation (15), provided the constants $C_m, \ldots, C_{m+l-1}$ are suitably chosen.

Example 4. $y'' - y = xe^x \quad (k_0 = 1,\ l = 2)$.

The characteristic equation $k^2 - 1 = 0$ has the roots $k_{1,2} = \pm 1$. The number $k_0 = 1$ is a root of the characteristic equation of multiplicity 1. Therefore, according to Lemma 2, we seek the particular solution in the form

$$y = (Ax + Bx^2)\, e^x.$$

We have

$$y' = [A + (A + 2B)\, x + Bx^2]\, e^x,$$

$$y'' = [2\,(A + B) + (A + 4B)\, x + Bx^2]\, e^x.$$

Substituting these expressions into the given equation we obtain

$$e^x\,(2A + 2B + 4Bx) = xe^x,$$

whence

$$2A + 2B = 0, \quad 4B = 1,$$

i.e.

$$A = -1/4, \quad B = 1/4.$$

Hence, the particular solution of our equation will be

$$y = (-x + x^2)\, e^x/4,$$

and the general solution

$$y = \left(-\frac{x}{4} + \frac{x^2}{4}\right) e^x + C_1 e^x + C_2 e^{-x}.$$

Remark. If the function y_i is a solution of the equation

$$L_n\,[y] = f_i\,(x) \quad (i = 1, \ldots, q),$$

then the function $y = \sum_{i=1}^{q} y_i$ is a solution of the equation

$$L_n\,[y] = L_n\left[\sum_{i=1}^{q} y_i\right] = \sum_{i=1}^{q} L_n\,[y_i] = \sum_{i=1}^{q} f_i\,(x).$$

From this equation and from what was said above it is clear that the form of the particular solution of the non-homogeneous equation $L_n\,[y] = f\,(x)$ with the right-hand member of the form

$$f\,(x) = P_{m-1}\,(x)\, e^{\alpha x} \sin \beta x + Q_{m-1}\,(x)\, e^{\alpha x} \cos \beta x,$$

where $P_{m-1}\,(x)$, $Q_{m-1}\,(x)$ are some algebraic polynomials of degree not higher than $(m - 1)$, coincides with the form

of the right-hand side if $\alpha \pm i\beta$ are not roots of the characteristic equation $R_n(k) = 0$.

If otherwise, i.e. if the numbers $\alpha \pm i\beta$ are the roots of the characteristic equation of multiplicity l, then the particular solution should also be sought in the form of the right-hand side, but the degrees of the algebraic polynomials must be raised by l unities.

Notice that if, for instance, the term $Q_{m-1}(x)\, e^{\alpha x} \cos \beta x$, is involved in the right-hand side, then the particular solution should be sought in the form of two terms

$$R_1(x)\, e^{\alpha x} \sin \beta x + R_2(x)\, e^{\alpha x} \cos \beta x,$$

where $R_1(x)$, $R_2(x)$ are polynomials of appropriate degrees.

Example 5. Write the form of the particular solution of the equation

$$y'' + y = x \sin x.$$

The characteristic equation $k^2 + 1 = 0$ has the roots $\pm i$. To the right-hand side of the given equation $x \sin x$ there corresponds the number $k_0 = \alpha \pm i\beta = 0 \pm i1 = \pm i$. These numbers are the roots of multiplicity 1 of the characteristic equation, therefore the particular solution should be sought in the form

$$y(x) = (Ax + Bx^2) \sin x + (Cx + Dx^2) \cos x.$$

Example 6. $y^{(4)} - 4y''' + 6y'' - 4y' + y = x^2 e^x$; the characteristic equation is $k^4 - 4k^3 + 6k^2 - 4k + 1 = (k-1)^4 = 0$. To the right-hand member $x^2 e^x$ there corresponds the number $\alpha = 1$ which is the root of the fourth multiplicity of the characteristic equation, hence, the particular solution should be sought in the form

$$y(x) = (Ax^4 + Bx^5 + Cx^6)\, e^x.$$

Sec. 1.19. Systems of Differential Equations. Phase Space

When studying the law of motion of a material point of mass m, it is convenient to use vector equations, i.e. equations written in the vector form. And so, let $\mathbf{r} = \mathbf{r}(t)$ be the law of motion of a material point in the space R^3, where t is time. This means that at an instant of time t the point

has a radius vector $\mathbf{r}(t)$ or, which is the same, the coordinates $\{x(t), y(t), z(t)\}$.

If a point of mass m moves under the action of a given force (vector) $\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$, then, according to Newton's law and mechanical meaning of the second derivative, the function $\mathbf{r}(t)$ must satisfy the equation of motion

$$m\ddot{\mathbf{r}} = \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}}). \tag{1}$$

Vector equation (1) is equivalent to the system of three scalar equations

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= X(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), \\ m\frac{d^2y}{dt^2} &= Y(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), \\ m\frac{d^2z}{dt^2} &= Z(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), \end{aligned}\right\} \tag{2}$$

where X, Y, Z are projections of the vector $\mathbf{F}$ on the coordinate axes.

If not only the coordinates of the point x, y, z, but also the projections of its velocity

$$\frac{d\mathbf{r}}{dt} = \{\dot{x}, \dot{y}, \dot{z}\}$$

are regarded as unknown, then we obtain a system of six first-order equations

$$\left.\begin{aligned} \dot{x} &= u, \quad m\frac{du}{dt} = X(t, x, y, z, u, v, w), \\ \dot{y} &= v, \quad m\frac{dv}{dt} = Y(t, x, y, z, u, v, w), \\ \dot{z} &= w, \quad m\frac{dw}{dt} = Z(t, x, y, z, u, v, w). \end{aligned}\right\} \tag{3}$$

Vector equation (1) can also be written in the form of a system of two vector equations if the velocity $\mathbf{V} = \frac{d\mathbf{r}}{dt}$ is regarded as an unknown vector function:

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}, \quad m\frac{d\mathbf{V}}{dt} = \mathbf{F}(t, \mathbf{r}, \mathbf{V}), \tag{1'}$$

where $\mathbf{V}$ is a vector with the projections u, v, w.

If we introduce into consideration the vector

$$\mathbf{R}(t) = \{x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)\},$$

then equation (1) or system (3) are equivalent to one vector equation of the first order

$$\frac{d\mathbf{R}}{dt} = \mathbf{\Phi}(t, x, y, z, u, v, w) \tag{4}$$

in the six-dimensional space and the vector

$$\mathbf{\Phi} = \left\{u, v, w, \frac{1}{m}X, \frac{1}{m}Y, \frac{1}{m}Z\right\}.$$

In physics, the six-dimensional space of points

$$(x, y, z, \dot{x}, \dot{y}, \dot{z}) \equiv (r_x, r_y, r_z, V_x, V_y, V_z)$$

is called the *phase space*, and the curve $\mathbf{R}(t)$ in the six-dimensional space, which is the solution of (4), is called a *phase trajectory*.

The phase space is a space of states of motion of a point along a curve.

The first three coordinates of $\mathbf{R}(t)$ characterize the position of a point in the three-dimensional space ($\mathbf{r}(t)$), while the remaining three coordinates of $\mathbf{R}(t)$ describe its velocity $\dot{\mathbf{r}}(t)$.

The above terminology presents a so-called *kinematic interpretation of a system of equations.*

System (3), or, which is the same, (4) is called a *dynamical system*.

To single out one trajectory it is necessary to specify the initial conditions $\mathbf{R}(t_0) = \mathbf{R}_0 = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$, i.e. the initial position of the point and its initial velocity. In other words, the integral curve $\mathbf{R}(t)$ must pass through the point $\mathbf{R}_0$ of the six-dimensional space.

Thus, solving physical problems leads us to the necessity of considering systems of differential equations.

Consider an arbitrary system of first-order differential equations of the form

$$\frac{dy_k}{dt} = f_k(t, y_1, \ldots, y_n) \quad (k = 1, \ldots, n), \tag{5}$$

where $y_k(t)$ are the required functions, and $f_k(t, y_1, \ldots, y_n)$ the known functions defined on some set of points $(t, y_1, \ldots, y_n)$ of the $(n+1)$-dimensional space.

We shall be interested in the solutions $y_1(t), \ldots, y_n(t)$ of system (5) which satisfy the initial conditions

$$y_1(t_0) = y_{10}, \ldots, y_n(t_0) = y_{n0}, \tag{6}$$

where $(y_{10}, \ldots, y_{n0})$ is a given point of the n-dimensional space.

System (5) (solved with respect to the derivatives of the sought functions!) is called normal (see Secs. 1.12 and 1.13).

If the functions f_k do not depend explicitly on the independent variable t, then system (5) is called an *autonomous normal system*

$$\dot{y}_k = f_k(y_1, \ldots, y_n) \quad (k = 1, \ldots, n). \tag{7}$$

With the following vectors introduced into n-dimensional space

$$\mathbf{y} = \{y_1(t), \ldots, y_n(t)\}, \quad \mathbf{y}_0 = \{y_{10}, \ldots, y_{n0}\},$$

$$\mathbf{F}(t, \mathbf{y}) = \{f_1(t, y_1, \ldots, y_n), \ldots, f_n(t, y_1, \ldots, y_n)\},$$

system (5) can be written in the form

$$\dot{\mathbf{y}} = \mathbf{F}(t, \mathbf{y}), \tag{5'}$$

and initial conditions (6) in the form

$$\mathbf{y}(t_0) = \mathbf{y}_0. \tag{6'}$$

The autonomous system can be written as follows:

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}), \quad \mathbf{F} = \{f_1(\mathbf{y}), \ldots, f_n(\mathbf{y})\}. \tag{7'}$$

An autonomous system can be interpreted in the following way: issued from every point $(y_1, \ldots, y_n)$ of a certain set of the n-dimensional space is the vector

$$\mathbf{F}(\mathbf{y}) = \{f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n)\}.$$

This defines a *vector field* on the mentioned set.

The solution $\mathbf{y}(t)$ describes a definite mechanical trajectory (path) of a point in the n-dimensional space, the veloc-

ity vector $\dot{\mathbf{y}}(t)$ coinciding with the vector $\mathbf{F}(\mathbf{y})$ at the instant when the point passes through $(y_1, \ldots, y_n)$.

An n-dimensional space of points $(y_1, \ldots, y_n)$ in which the solutions of autonomous system (7′) are interpreted in the form of trajectories is called the *phase space of the system*

The trajectories $\mathbf{y}(t)$ are called *phase trajectories*, an the vectors $\mathbf{F}(\mathbf{y})$ *phase velocities.*

The problem on existence of the solution of a normal system was considered in Sec. 1.12.

Sec. 1.20. Homogeneous Linear System of Differential Equations

The system

$$\left.\begin{aligned} \frac{dy_1}{dt} &= a_{11}(t)\, y_1 + \ldots + a_{1n}(t)\, y_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dy_n}{dt} &= a_{n1}(t)\, y_1 + \ldots + a_{nn}(t)\, y_n, \end{aligned}\right\} \qquad (1)$$

where the coefficients $a_{kl}(t)$ are continuous functions (or constant numbers) defined in the interval (a, b), is called a *homogeneous linear system of first-order differential equations.*

For the sake of notational convenience, we shall use vector-matrix notation for representing systems of equations. Let us introduce the matrix of the coefficients of system (1)

$$A(t) = \begin{pmatrix} a_{11}(t) & \ldots & a_{1n}(t) \\ \cdots & \cdots & \cdots \\ a_{n1}(t) & \ldots & a_{nn}(t) \end{pmatrix}$$

and write the sought-for system of functions in the form of a one-column matrix

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ \cdots \\ y_n(t) \end{pmatrix} = \text{colon } [y_1, \ldots, y_n].$$

Let us now introduce some general concepts concerning matrices, and let for the time being $A(t)$ be an arbitrary matrix.

If the functions $a_{kl}(t)$ are continuously differentiable, then it is possible to introduce the notion of the *derivative of a matrix* meaning the matrix consisting of the derivatives of the elements of the original matrix. Thus,

$$A'(t) = \frac{dA(t)}{dt} = \dot{A}(t) = \begin{pmatrix} \dot{a}_{11}(t) & \dots & \dot{a}_{1n}(t) \\ \dots & \dots & \dots \\ \dot{a}_{n1}(t) & \dots & \dot{a}_{nn}(t) \end{pmatrix}.$$

The following properties are readily verified:

(1) If C is a constant matrix, the $\frac{dC}{dt} = 0$, where 0 is a null matrix, all of its elements being zeros.

(2) If the matrices $A(t)$ and $B(t)$ are of one and the same order (dimension), then

$$\frac{d}{dt}[A(t) + B(t)] = \dot{A}(t) + \dot{B}(t).$$

(3) If for the matrices $A(t)$ and $B(t)$ multiplication is feasible then

$$\frac{d}{dt}[A(t)\,B(t) = \dot{A}(t)\,B(t) + A(t)\,\dot{B}(t).$$

(4) Let $A^{-1}(t)$ be the inverse of $A(t)$, i.e.

$$A^{-1}(t)\,A(t) = A(t)\,A^{-1}(t) = E,$$

where E is a unit matrix. Differentiating this equality we obtain

$$\dot{A}(t)\,A^{-1}(t) + A(t)\,\frac{d}{dt}A^{-1}(t) = 0,$$

$$A(t)\,\frac{d}{dt}A^{-1}(t) = -\dot{A}(t)\,A^{-1}(t),$$

$$\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\,\dot{A}(t)\,A^{-1}(t).$$

Analogously, we introduce the notion of the *integral of a matrix*. This is a matrix whose elements are integrals of the elements of the original matrix:

$$\int_{t_0}^{t} A(\tau)\,d\tau = \left(\int_{t_0}^{t} a_{kl}(\tau)\,d\tau\right), \quad t_0,\ t \in [a,\ b].$$

This notion possesses the properties similar to ordinary properties of integrals of functions. We shall note here only one property—the analogue of the Newton-Leibniz* formula.

If $F(t) = \Phi'(t)$, then

$$\int_{t_0}^{t} F(\tau)\, d\tau = \Phi(t) - \Phi(t_0).$$

According to the rule for multiplying matrices, system (1) can be written as

$$\frac{d\mathbf{y}}{dt} = A(t)\,\mathbf{y}. \tag{1'}$$

Let us recall that by equality of matrices we understand the equality of their corresponding elements.

Let us consider some obvious properties.

If two vector functions $\mathbf{y}$ and $\mathbf{z}$ satisfy system (1'), i.e.

$$\frac{d\mathbf{y}}{dt} = A(t)\,\mathbf{y}, \quad \frac{d\mathbf{z}}{dt} = A(t)\,\mathbf{z},$$

then their sum also satisfy system (1'):

$$\frac{d(\mathbf{y}+\mathbf{z})}{dt} = \frac{d\mathbf{y}}{dt} + \frac{d\mathbf{z}}{dt} = A(t)\,\mathbf{y} + A(t)\,\mathbf{z} = A(t)\,(\mathbf{y}+\mathbf{z}). \tag{2}$$

Besides, if c is a number, then the vector function $c\mathbf{y}$ satisfies system (1'):

$$\frac{d(c\mathbf{y})}{dt} = c\,\frac{d\mathbf{y}}{dt} = cA(t)\,\mathbf{y} = A(t)\,(c\mathbf{y}). \tag{3}$$

From these properties it follows by induction that if

$$\mathbf{y}^1(t) = \text{colon}\,[y_{11}(t), \ldots, y_{n1}(t)], \ \ldots, \ \mathbf{y}^n(t) = \text{colon}\,[y_{1n}(t), \ldots, y_{nn}(t)] \tag{4}$$

are solutions of system (1') and $C_1, \ldots, C_n$ are arbitrary numbers, then

$$\mathbf{y}(t) = C_1\mathbf{y}^1(t) + \ldots + C_n\mathbf{y}^n(t) \tag{5}$$

is a solution of (1').

* Newton, Isaac (1642-1727). Great English mathematician, physicist, and astronomer.

Leibniz, Gottfried Wilhelm (1646-1716). Great German mathematician and philosopher.

It is of importance to note that *if system* (4) *of solutions is linearly independent, then formula* (5) *expresses the general solution of system* (1′); *in other words, for different constants* $C_1, \ldots, C_n$ *formula* (5) *contains all possible solutions of system* (1′).

This statement can be proved much in the same manner as in the case of a linear equation of the nth order.

System (4) of vector functions is said to be linearly independent on (a, b) if from the equality

$$C_1\mathbf{y}^1(t) + \ldots + C_n\mathbf{y}^n(t) \equiv \mathbf{0} \quad (a < t < b) \tag{6}$$

it follows that $C_1 = \ldots = C_n = 0$.

Since vector equation (6) is equivalent to n scalar equalities

$$\left.\begin{aligned} C_1y_{11}(t) + \ldots + C_ny_{1n}(t) &= 0. \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ C_1y_{n1}(t) + \ldots + C_ny_{nn}(t) &= 0, \end{aligned}\right\} \tag{6′}$$

then from the fact that the determinant

$$W(t) = |y_{kl}(t)| \tag{7}$$

is not zero at least for one value of t, already follows, obviously, linear independence of system (4) of vector functions.

The determinant $W(t)$ is called the *Wronskian of system* (4) *of vector functions.*

If system of vectors (4) is linearly independent on (a, b) and these vectors are solutions of system (1′) with continuous coefficients, then it is possible to prove that $W(t) \neq 0$ for all $t \in (a, b)$.

Thus, the *condition* $W(t) \neq 0$ *for all* $t \in (a, b)$ *is a necessary and sufficient condition for solutions* (4) *of system* (1) *with continuous coefficients to be linearly independent on* (a, b).

Hence, to obtain the *general solution of homogeneous system* (1), *we have to find n linearly independent solutions* (4) *of system* (1). *Sum* (5), *where* $C_1, \ldots, C_n$ *are arbitrary constants, is the general solution of system* (1).

Notice that the solutions $y_1(t), \ldots, y_n(t)$ of system (1) exist on the same interval (a, b), where the coefficients $a_{kl}(t)$ of system (1) are defined and continuous.

The system of n linearly independent on (a, b) solutions of system (1) is usually called a *fundamental system of solutions* of system (1). This system can be characterized by the square matrix

$$Y(t) = \begin{pmatrix} y_{11}(t) & \dots & y_{1n}(t) \\ \dots & \dots & \dots \\ y_{n1}(t) & \dots & y_{nn}(t) \end{pmatrix},$$

which is called the *fundamental matrix of system* (1).

Thus, in a fundamental matrix solutions (4) are arranged in columns. Note that in the notation $y_{jk}(t)$ the first index j denotes the number of coordinate, and the second, k, the number of solution.

Let us show that the fundamental matrix $Y(t)$ satisfies the matrix equation

$$\dot{Y}(t) = A(t)\ Y(t). \tag{8}$$

Since the functions $y_{sk}(t)$ satisfy the jth equation of system (1), then

$$\frac{dy_{jk}(t)}{dt} = \sum_{s=1}^{n} a_{js}(t)\, y_{sk}(t).$$

Consequently, according to the rule for multiplying matrices (in this case square ones), we have

$$\dot{Y}(t) = \left(\frac{dy_{jk}(t)}{dt}\right) = \left(\sum_{s=1}^{n} a_{js}(t)\, y_{sk}(t)\right) = A(t)\, Y(t),$$

which was to be proved.

Conversely, if the matrix $Y(t) = (y_{jk}(t))$ satisfies matrix equation (8), then its columns

$$\mathbf{y}^k(t) = \text{colon}\,[y_{1k}(t), \dots, y_{nk}(t)] \quad (k = 1, \dots, n)$$

represent solutions of homogeneous linear system (1).

And if

$$|\,Y(t)\,| = W(t) \neq 0,$$

then the matrix $Y(t)$ is a fundamental one.

Indeed,

$$\mathbf{y}^k(t) = Y(t)\,\mathbf{e}_k,$$

where $\mathbf{e}_k$ = colon $[0, \ldots, 0, 1, 0, \ldots, 0]$. Multiplying equation (8) by $\mathbf{e}_k$, we obtain

$$\frac{dY(t)}{dt}\mathbf{e}_k = \frac{d}{dt}[Y(t)\,\mathbf{e}_k] = A(t)\,[Y(t)\,\mathbf{e}_k],$$

i.e.

$$\frac{d}{dt}\mathbf{y}^k(t) = A(t)\,\mathbf{y}^k(t) \quad (k = 1, \ldots, n).$$

If $Y(t)$ is the fundamental matrix of system (1), then the general solution (5) of system (1) can be written briefly in the form

$$\mathbf{y}(t) = Y(t)\,\mathbf{C}, \tag{9}$$

where $\mathbf{C}$ = colon $[C_1, \ldots, C_n)$ is a constant column matrix with arbitrary elements.

Setting in identity (9) $t = t_0$, we have

$$\mathbf{y}(t_0) = Y(t_0)\,\mathbf{C}.$$

Hence

$$\mathbf{C} = Y^{-1}(t_0)\,\mathbf{y}(t_0).$$

Consequently,

$$\mathbf{y}(t) = Y(t)\,Y^{-1}(t_0)\,\mathbf{y}(t_0).$$

The matrix

$$Y(t)\,Y^{-1}(t_0) = K(t, t_0)$$

is termed *Cauchy's matrix*.

With the aid of this matrix, the solution of system (1) can be written as follows:

$$\mathbf{y}(t) = K(t, t_0)\,\mathbf{y}(t_0). \tag{10}$$

In particular, if the fundamental matrix $Y(t)$ is normed for $t = t_0$, i.e. $Y(t_0) = Y^{-1}(t_0) = E$, where E is a unit matrix, then formula (10) takes the form

$$\mathbf{y}(t) = Y(t)\,\mathbf{y}(t_0). \tag{11}$$

Sec. 1.21. General Solution of a Homogeneous Linear System of Differential Equations with Constant Coefficients

We shall seek the solution of the homogeneous linear system with the constant coefficients a_{kl}

$$\left.\begin{aligned} \frac{dy_1}{dt} &= a_{11}y_1 + \ldots + a_{1n}y_n, \\ &\cdots\cdots\cdots\cdots\cdots \\ \frac{dy_n}{dt} &= a_{n1}y_1 + \ldots + a_{nn}y_n, \end{aligned}\right\} \quad -\infty < t < \infty, \tag{1}$$

or

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}, \tag{1'}$$

where $A = \| a_{kl} \|$ is a numerical matrix given in the following form:

$$y_1 = \alpha_1 e^{\lambda t}, \quad y_2 = \alpha_2 e^{\lambda t}, \quad \ldots, \quad y_n = \alpha_n e^{\lambda t} \tag{2}$$

or

$$\mathbf{y} = \{\alpha_1 e^{\lambda t}, \quad \ldots, \quad \alpha_n e^{\lambda t}\}. \tag{2'}$$

The numbers $\alpha_1, \ldots, \alpha_n, \lambda$ are to be determined. Of course, the numbers

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad \ldots, \quad \alpha_n = 0$$

yield a trivial solution of system (1):

$$y_1(t) \equiv 0, \quad \ldots, \quad y_n(t) \equiv 0.$$

But we are interested in the nontrivial solutions corresponding to the nonzero vectors

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n).$$

We have

$$\frac{dy_i}{dt} = \alpha_i \lambda e^{\lambda t} \quad (i = 1, \ldots, n).$$

Substituting the functions $y_i(t)$ and their derivatives into (1), reducing by $e^{\lambda t}$ ($e^{\lambda t} \neq 0$) and transposing the terms,

we obtain

$$\left.\begin{array}{l} (a_{11}-\lambda)\alpha_1 + a_{12}\alpha_2 + \ldots + a_{1n}\alpha_n = 0, \\ a_{21}\alpha_2 + (a_{22}-\lambda)\alpha_2 + \ldots + a_{2n}\alpha_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots + (a_{nn}-\lambda)\alpha_n = 0, \end{array}\right\} \quad (3)$$

or in the matrix form

$$(A - \lambda E)\,\boldsymbol{\alpha} = \mathbf{0}, \quad (3')$$

or

$$A\boldsymbol{\alpha} = \lambda\boldsymbol{\alpha}, \quad (4)$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, and

$$E = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

is a unit matrix.

For system (3) to have a nontrivial solution, it is necessary and sufficient that its determinant be equal to zero:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0. \quad (5)$$

Equation (5) is referred to as the characteristic equation of system (1). From equation (5) we just find the values of λ for which system (4) has nontrivial solutions $\boldsymbol{\alpha}$.

The left-hand member of (5) is a polynomial of degree n with respect to the variable λ. Taking into account the multiplicity, this polynomial has n roots:

$$\lambda_1, \lambda_2, \ldots, \lambda_n. \quad (6)$$

If all the n zeros are different, then, substituting each of them λ_j into system (4) and solving the latter, we obtain a certain nontrivial vector

$$\boldsymbol{\alpha}^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_n^{(j)}) \quad (7)$$

which satisfies this system. This vector is defined not identically, with an accuracy to a scalar factor.

From (4) it is seen that λ_j are eigenvalues of the matrix (linear transformation) A, and $\alpha^{(j)}$ the eigenvectors of A.

The vectors $\alpha^{(j)}$ $(j = 1, \ldots, n)$ are linearly independent if all the eigenvalues are distinct. This can be proved by induction.

Let us prove that any k vectors of this system are linearly independent of one another.

For $k = 1$ this is obvious, since each of the vectors $\alpha^{(j)}$ is nontrivial. Let our assertion be true for $k - 1$ vectors and let us prove it for k vectors.

Let us assume the contrary. Let, for instance, the first k vectors of our system be linearly dependent. Then

$$\sum_{j=1}^{k} C_j \alpha^{(j)} = 0, \tag{8}$$

where at least one of the coefficients C_j is different from zero. For the sake of definiteness, we shall assume that $C_1 \neq 0$. We then apply the linear transformation generated by the matrix A to both sides of (8):

$$A\left(\sum_{j=1}^{k} C_j \alpha^{(j)}\right) = \sum_{j=1}^{k} C_j \lambda_j \alpha^{(j)} = 0, \tag{9}$$

Multiplying (8) by λ_k and subtracting then from (9), we obtain

$$C_1 (\lambda_1 - \lambda_k) \alpha^{(1)} + C_2 (\lambda_2 - \lambda_k) \alpha^{(2)} + \ldots + C_{k-1} (\lambda_{k-1} - \lambda_k) \alpha^{(k-1)} = 0,$$

where $\lambda_i - \lambda_k \neq 0$ for all $i \neq k$.

Hence, $C_1 (\lambda_1 - \lambda_k) \neq 0$ and we have obtained that the vectors $\alpha^{(1)}, \ldots, \alpha^{(k-1)}$ are linearly dependent which contradicts out assumption.

Thus, the eigenvectors $\alpha^{(j)}$ $(j = 1, \ldots, n)$ responding to distinct eigenvalues λ_j are linearly independent, and the determinant formed from the coordinates of these vectors is not equal to zero:

$$|\alpha_k^{(j)}| \neq 0.$$

Remark 1. Since in the n-dimensional space there cannot be more than n linearly independent vectors, then to each eigenvalue λ_j (provided all of them are distinct!) there

corresponds only one eigenvector with an accuracy to a constant factor.

As a result, we obtain n vector functions which are solutions of system (1)

$$\left.\begin{aligned} &\mathbf{y}^1(t) = \{\alpha_1^{(1)} e^{\lambda_1 t},\ \alpha_2^{(1)} e^{\lambda_1 t},\ \ldots,\ \alpha_n^{(1)} e^{\lambda_1 t}\},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\mathbf{y}^n(t) = \{\alpha_1^{(n)} e^{\lambda_n t},\ \alpha_2^{(n)} e^{\lambda_n t},\ \ldots,\ \alpha_n^{(n)} e^{\lambda_n t}\}\\ &\qquad (-\infty < t < \infty). \end{aligned}\right\} \tag{10}$$

Vector functions (10) form a linearly independent system on the interval $(-\infty, \infty)$, since their Wronskian

$$W(t) = \exp(\lambda_1 t + \ldots + \lambda_n t) \cdot |\alpha_k^{(j)}| \neq 0.$$

Therefore the general solution of system (1) has the form

$$\mathbf{y} = \sum_{j=1}^{n} C_j \mathbf{y}^j(t), \tag{11}$$

where C_j are arbitrary constants. The general solution can be written in the expanded form:

$$\left.\begin{aligned} &y_1(t) = \sum_{j=1}^{n} C_j \alpha_1^{(j)} \exp(\lambda_j t),\\ &\cdots\cdots\cdots\cdots\\ &y_n(t) = \sum_{j=1}^{n} C_j \alpha_n^{(j)} \exp(\lambda_j t). \end{aligned}\right\} \tag{11'}$$

Remark 2. In practice, instead of finding the vectors $\boldsymbol{\alpha}^{(j)} = (\alpha_1^{(j)}, \ldots, \alpha_n^{(j)})$ from linear system (4), we usually seek for the general solution of system (1) in the form

$$\left.\begin{aligned} &y_1(t) = \sum_{j=1}^{n} a_1^{(j)} \exp(\lambda_j t),\\ &\cdots\cdots\cdots\cdots\\ &y_n(t) = \sum_{j=1}^{n} a_n^{(j)} \exp(\lambda_j t), \end{aligned}\right\}$$

where $\lambda_1, \ldots, \lambda_n$ are the roots (distinct!) of the characteristic equation, and $a_k^{(j)}$ are the numbers to be found. From the above stated theory it follows that such numbers exist.

To find the numbers $a_k^{(j)}$, we substitute the functions $y_j(t)$ into system (1) and compare the coefficients of the like terms $\exp(\lambda_j t)$. In this case we fail to obtain unique numbers $a_k^{(j)}$, since they depend on n arbitrary constants.

Example 1. Solve the system

$$\frac{dx}{dt} = x + 2y,$$
$$\frac{dy}{dt} = y + 2x.$$

The characteristic equation

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0, \quad (\lambda-1)^2 - 4 = 0,$$

has the roots $\lambda_1 = 3$, $\lambda_2 = -1$. Proceeding from the structure of the general solution, we shall seek the solution in the form

$$x(t) = C_1 e^{3t} + C_2 e^{-t}, \quad y(t) = a_1 e^{3t} + a_2 e^{-t}$$

(i.e. we omit the process of finding the numbers $\alpha_k^{(j)}$).

We express the coefficients a_1, a_2 in terms of C_1, C_2. Substituting the functions $x(t)$ and $y(t)$ into one of the equations of the system, we obtain

$$3C_1 e^{3t} - C_2 e^{-t} = C_1 e^{3t} + C_2 e^{-t} + 2a_1 e^{3t} + 2a_2 e^{-t},$$

whence

$$2C_1 - 2a_1 = 0, \quad -2C_2 - 2a_2 = 0, \text{ i.e.}$$
$$a_1 = C_1, \quad a_2 = -C_2.$$

Thus, the general solution has the form

$$x(t) = C_1 e^{3t} + C_2 e^{-t},$$
$$y(t) = C_1 e^{3t} - C_2 e^{-t}.$$

Let now characteristic equation (5) of system (1) have the root λ_1 of multiplicity r. Let us reduce system (1) to one equation of order n with respect to the function $y_1(t)$. This equation and system (1) have one and the same characteristic equation (the proof is given below). But then, as we know, to the root λ_1 of multiplicity r there corresponds

the solution of the equation of the nth order of the form

$$y_1(t) = (b_0 + b_1 t + \ldots + b_{r-1} t^{r-1}) e^{\lambda_1 t},$$

where $b_0, b_1, \ldots, b_{r-1}$ are arbitrary constants. Thus

$$y_1(t) = P_{r-1,1}(t) e^{\lambda_1 t},$$

where $P_{r-1,1}(t)$ is a polynomial of degree $r - 1$.

Reasoning in a similar manner, we can also express other functions $y_j(t)$ in the form

$$y_j(t) = P_{r-1,j}(t) e^{\lambda_1 t} \quad (j = 1, \ldots, n), \tag{12}$$

where $P_{r-1,j}(t)$ are polynomials of degree $r - 1$.

Each of the functions $y_j(t)$ satisfies its differential equation of the nth order, whatever the coefficients of the polynomial $P_{r-1,j}(t)$ may be.

It remains to choose from among the polynomials $P_{r-1,1}, \ldots, P_{r-1,n}$ such ones that their corresponding functions $y_1(t), \ldots, y_n(t)$ satisfy simultaneously system (1). To this end it is necessary to substitute y_j into system (1), reduce by $e^{\lambda_1 t}$, and compare the coefficients of like powers of t. The sought-for coefficients will depend on r arbitrary constants. Sometimes the polynomial $P_{r-1,1}(t)$ may be recommended to be taken arbitrarily, then the coefficients of the remaining polynomials $P_{r-1,j}(t)$ $(j = 2, \ldots, n)$ will be defined indentically in terms of the coefficients $P_{r-1,1}$.

But it may happen that in doing so we shall come to a contradiction showing that in the present case some coefficients of the polynomial $P_{r-1,1}$ are actually equal to zero and therefore they cannot be regarded as arbitrary.

Other multiple roots of the characteristic equation (if any) are considered much in the same way. The solutions corresponding to the simple roots should be sought for in the form (8), as it was explained above.

To obtain the general solution of system (1), we have to take the sum of the indicated solutions (vector functions).

In simple cases the solution may be sought directly in the form of the sum of similar solutions. This is best of all illustrated by particular examples.

Example 2. Solve the system

$$\frac{dy_1}{dt} = y_1 - y_2,$$

$$\frac{dy_2}{dt} = y_1 + 3y_2.$$

The characteristic equation $\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$ or $\lambda^2 - 4\lambda + 4 = 0$ has the multiple root $\lambda_1 = 2$.

The differential equation corresponding to our system for the function y_1 has the form $\ddot{y}_1 - 4\dot{y}_1 + 4y_1 = 0$. It has the same characteristic equation.

The solution of the system should be sought in the form

$$y_1(t) = (C_1 + C_2 t)\, e^{2t}, \quad y_2(t) = (a_1 + a_2 t)\, e^{2t}.$$

Substituting these functions into the system, we obtain

$$C_2 + 2C_1 + 2C_2 t = C_1 + C_2 t - a_1 - a_2 t,$$

$$a_2 + 2a_1 + 2a_2 t = C_1 + C_2 t + 3(a_1 + a_2 t).$$

Equating the coefficients of like powers of t (in the first equality), we get $a_1 = -C_1 - C_2$, $a_2 = -C_2$. The second equality yields the same solution. Hence, the general solution of the system has the form

$$y_1(t) = (C_1 + C_2 t)\, e^{2t}, \quad y_2(t) = -(C_1 + C_2 + C_2 t)\, e^{2t}.$$

Example 3. Solve the system

$$\dot{y}_1 = 2y_1 + y_2 + y_3,$$

$$\dot{y}_2 = y_1 + 2y_2 + y_3,$$

$$\dot{y}_3 = y_1 + y_2 + 2y_3.$$

The characteristic equation has the form

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

or

$$(1-\lambda)^2 (\lambda - 4) = 0.$$

Thus, $\lambda_1 = 1$ is a root of the second multiplicity, and $\lambda_3 = 4$ is a simple root of the characteristic equation. For $\lambda_1 = 1$

system (3) has the form

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0, \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0, \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0. \end{aligned}$$

Therefore α_1, α_2, α_3 must be such that $\alpha_1 + \alpha_2 + \alpha_3 = 0$, i.e. we have two free variables, say, α_2 and α_3.

Therefore, for instance, $\boldsymbol{\alpha}^{(1)} = (-1, 1, 0)$, $\boldsymbol{\alpha}^{(2)} = (-1, 0, 1)$ are linearly independent eigenvectors of matrix A of our system (matrix A is a symmetric one). For $\lambda_3 = 4$ we find $\boldsymbol{\alpha}^{(3)} = (1, 1, 1)$. Therefore

$$\mathbf{y}^1 = \begin{pmatrix} -e^t \\ e^t \\ 0 \end{pmatrix}, \quad \mathbf{y}^2 = \begin{pmatrix} -e^t \\ 0 \\ e^t \end{pmatrix}, \quad \mathbf{y}^3 = \begin{pmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{pmatrix}$$

are linearly independent solutions and

$$\mathbf{y} = C_1\mathbf{y}^1 + C_2\mathbf{y}^2 + C_3\mathbf{y}^3$$

is the general solution of our system, which can be written in the expanded form:

$$\begin{aligned} y_1(t) &= -(C_1 + C_2)e^t + C_3e^{4t} \\ y_2(t) &= C_1e^t + C_3e^{4t}, \\ y_3(t) &= C_2e^t + C_3e^{4t}. \end{aligned}$$

Remark 3. If matrix A is a *symmetric* one ($a_{kl} = a_{lk}$), then the corresponding linear operator A will be *self-adjoint*, and, as we know, in this case, whatever roots (including multiple ones) equation (5) may have, there exists a system of n linearly independent eigenvectors $\boldsymbol{\alpha}$ and, consequently, knowing the vectors $\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)}$*, we can write the general solution of system (1) with a symmetric matrix A.

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* See our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Secs. 15, 22. 25.

Sec. 1.22. Reducing a System of Equations to One Equation

Lemma 1. *Let there be given a linear system of first-order differential equations with constant coefficients* a_{kl}

$$\left.\begin{aligned} \frac{dy_1}{dt} &= a_{11}y_1 + \ldots + a_{1n}y_n - f_1(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dy_n}{dt} &= a_{n1}y_1 + \ldots + a_{nn}y_n - f_n(t), \end{aligned}\right\} \tag{1}$$

where $f_j(t)$ *are functions defined on* (a, b) *and having a necessary number of derivatives.*

If the functions $y_1(t), \ldots, y_n(t)$ *satisfy this system, then the functions* $y_j(t)$ *satisfies the following differential equation of the nth order:*

$$D\left(\frac{d}{dt}\right) y_j(t) = \Phi_j(t) \quad (j = 1, \ldots, n),$$

where

$$D(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix} \tag{2}$$

is the characteristic polynomial of system (1) *and the functions* $\Phi_j(t)$ *are defined in terms of* a_{kl} and f_s $(s = 1, \ldots, n)$ (see below (4)).

Proof. Let us write system (1) in the following form:

$$\left.\begin{aligned} \left(a_{11} - \frac{d}{dt}\right) y_1 + a_{12}y_2 + \ldots + a_{1n}y_n &= f_1(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}y_1 + a_{n2}y_2 + \ldots + \left(a_{nn} - \frac{d}{dt}\right) y_n &= f_n(t). \end{aligned}\right\} \tag{1'}$$

The element in row k and column l of determinant (2) will be denoted by $b_{kl}(\lambda)$, and the corresponding cofactor by $M_{kl}(\lambda)$ (for instance, $b_{11}(\lambda) = a_{11} - \lambda$, $b_{12}(\lambda) = a_{12}$). Substituting $\frac{d}{dt}$ for λ in these expressions, we obtain differential

operators $b_{kl}\left(\frac{d}{dt}\right)$, $M_{kl}\left(\frac{d}{dt}\right)$. To them we may apply the arguments used in proving Kronecker's theorem*

We have

$$\sum_{s=1}^{n} b_{si}(\lambda) M_{sj}(\lambda) = \delta_{ij} \cdot D(\lambda), \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$$

(i.e. the sum of the products of the elements in some column of the determinant $D(\lambda)$ by the cofactors of the elements of the same column (other column) is equal to the determinant $D(\lambda)$ (is equal to zero). Therefore

$$\sum_{s=1}^{n} b_{si}\left(\frac{d}{dt}\right) M_{sj}\left(\frac{d}{dt}\right) = \delta_{ij} \cdot D\left(\frac{d}{dt}\right). \tag{3}$$

We now apply the operator $M_{1j}\left(\frac{d}{dt}\right)$ to the first equation of (1′) (or, which is more precise, to both of its members) (that is, we perform a number of differentiations, multiplications by a number, and additions), the operator $M_{2j}\left(\frac{d}{dt}\right)$ to the second equation of (1′) and so forth, and add the results. Then, on the basis of (3), we obtain

$$\sum_{s=1}^{n} M_{sj}\left(\frac{d}{dt}\right) b_{sj}\left(\frac{d}{dt}\right) y_j = D\left(\frac{d}{dt}\right) y_j = \sum_{s=1}^{n} M_{sj}\left(\frac{d}{dt}\right) f_s(t),$$

i.e.

$$D\left(\frac{d}{dt}\right) y_j = \Phi_j(t) \quad (j = 1, \ldots, n),$$

where

$$\Phi_j(t) = \sum_{s=1}^{n} M_{sj}\left(\frac{d}{dt}\right) f_s(t). \tag{4}$$

Remark 1. If all $f_s(t) \equiv 0$ $(s = 1, \ldots, n)$, then $\Phi_j(t) \equiv \equiv 0$ and we shall have one and the same differential equation for all the functions $y_j(t)$:

$$D\left(\frac{d}{dt}\right) y_j = 0.$$

* See our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 4.

Example 1. Reduce the system

$$\dot{y}_1 = y_1 + y_2,$$
$$\dot{y}_2 = \quad y_2$$

to one equation. Let us write this system in the form

$$\left.\begin{aligned}\left(1-\frac{d}{dt}\right)y_1+y_2=0,\\ 0+\left(1-\frac{d}{dt}\right)y_2=0.\end{aligned}\right\} \tag{5}$$

Here $b_{11}=1-\frac{d}{dt}$, $b_{12}=1$, $b_{21}=0$, $b_{22}=1-\frac{d}{dt}$; $M_{11}=1-\frac{d}{dt}$, $M_{12}=0$, $M_{21}=-1$, $M_{22}=1-\frac{d}{dt}$. Applying operator M_{11} to the first equation of (5), operator M_{21} to the second, and adding up, we get

$$\left(1-\frac{d}{dt}\right)^2 y_1+\left(1-\frac{d}{dt}\right)y_2-\left(1-\frac{d}{dt}\right)y_2=0,$$

i.e.

$$\left(1-\frac{d}{dt}\right)^2 y_1=0, \quad \ddot{y}_1-2\dot{y}_1+y_1=0.$$

Quite analogously, applying operator $M_{12}=0$ to the first equation of (5), operator $M_{22}=1-\frac{d}{dt}$ to the second, and adding up, we obtain

$$\left(1-\frac{d}{dt}\right)^2 y_2=0.$$

This differential equation can be obtained in a different way. For instance, differentiating the second equation, we have

$$\ddot{y}_2=\dot{y}_2.$$

Subtracting the second equation of the system from this equation, we obtain

$$\ddot{y}_2-2\dot{y}_2+y_2=0.$$

Sec. 1.23. Nonhomogeneous System of Linear Differential Equations with Constant Coefficients

The system of equations

$$\left.\begin{aligned} \frac{dy_1}{dt} &= a_{11}y_1 + \ldots + a_{1n}y_n + f_1(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dy_n}{dt} &= a_{n1}y_1 + \ldots + a_{nn}y_n + f_n(t), \end{aligned}\right\} \tag{1}$$

or

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} + \mathbf{f}(t), \tag{1'}$$

where $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t))$ is a given vector function continuous on (a, b), a_{kl} are given constant numbers, and $A = \| a_{kl} \|$, is called a *nonhomogeneous system of first-order differential equations with constant coefficients.*

The general solution of system (1) *is equal to the sum*

$$\mathbf{y}(t) = \sum_{k=1}^{n} C_k \mathbf{y}^k(t) + \mathbf{y}^0(t) \tag{2}$$

of some of its particular solution $\mathbf{y}^0(t)$ *and the general solution* $\sum_{k=1}^{n} C_k \mathbf{y}^k(t)$ *of the corresponding homogeneous system*

$$L[\mathbf{y}] \equiv \frac{d\mathbf{y}}{dt} - A\mathbf{y} = \mathbf{0}. \tag{3}$$

Indeed, for any constants C_k sum (2) is obviously a solution of system (1):

$$L\left[\sum_{k=1}^{n} C_k \mathbf{y}^k + \mathbf{y}^0\right] = L[\mathbf{y}^0] = \mathbf{f}.$$

On the other hand, if $\mathbf{y}$ is a solution of system (1), then

$$L[\mathbf{y} - \mathbf{y}^0] = L[\mathbf{y}] - L[\mathbf{y}^0] = \mathbf{f} - \mathbf{f} = \mathbf{0},$$

but then for some constants C_k

$$\mathbf{y} - \mathbf{y}^0 = \sum_{k=1}^{n} C_k \mathbf{y}^k.$$

If the general solution of homogeneous system (3) is known, then the particular solution of nonhomogeneous system (1) can be found using the method of variation of arbitrary constants (*Lagrange's* method*).

Let

$$\mathbf{y}(t) = \sum_{k=1}^{n} C_k \mathbf{y}^k(t)$$

be the general solution of system (3), i.e. $\mathbf{y}^k(t)$ are linearly independent particular solutions of (3):

$$L[\mathbf{y}^k] = \mathbf{0} \quad (k = 1, \ldots, n).$$

We shall regard $C_k = C_k(t)$ as functions of t and choose them so that the function

$$\mathbf{U}(t) = \sum_{k=1}^{n} C_k(t)\, \mathbf{y}^k(t) \tag{4}$$

is a particular solution of nonhomogeneous system (1). Differentiating, we have

$$\frac{d\mathbf{U}}{dt} = \sum_{k=1}^{n} \left[C_k(t) \frac{d\mathbf{y}^k}{dt} + C_k'(t)\, \mathbf{y}^k(t) \right].$$

Substituting the values of $\mathbf{U}$ and $\frac{d\mathbf{U}}{dt}$ into (1′), we obtain

$$\sum_{k=1}^{n} \left[C_k(t) \frac{d\mathbf{y}^k}{dt} + C_k'(t)\, \mathbf{y}^k(t) \right] - \sum_{k=1}^{n} C_k(t)\, A\mathbf{y}^k = \mathbf{f}(t),$$

or

$$\sum_{k=1}^{n} C_k(t)\, L[\mathbf{y}^k] + \sum_{k=1}^{n} C_k'(t)\, \mathbf{y}^k(t) = \mathbf{f}(t).$$

Since $L[\mathbf{y}^k] = \mathbf{0}$, for determining $C_k'(t)$ we obtain the system

$$\sum_{k=1}^{n} C_k'(t)\, \mathbf{y}^k(t) = \mathbf{f}(t) \tag{5}$$

with the vector functions $\mathbf{f}(t)$ and $\mathbf{y}^k(t)$ continuous on (a, b).

* Lagrange, Joseph Louis (1736-1813). Outstanding French physicist and mathematician.

System (5) is a linear one with respect to $C_k'(t)$ with a determinant equal to the Wronskian of the system of vectors $\mathbf{y}^1, \ldots, \mathbf{y}^n$. Since this determinant is not equal to zero, system (5) has a unique solution: $C_k'(t) = \varphi_k(t)$ $(k = 1, \ldots, n)$. The functions $\varphi_k(t)$ are continuous due to the continuity of the vector functions $\mathbf{f}(t)$ and $\mathbf{y}^k(t)$.

Integrating, we find

$$C_k(t) = \int \varphi_k(t)\, dt \quad (k = 1, \ldots, n).$$

Substituting into (4), we obtain the particular solution of system (1).

Example 1. Solve the system

$$\dot{y}_1 = y_1 + 2y_2 + e^t,$$
$$\dot{y}_2 = 2y_1 + y_2.$$

It is readily verified that

$$y_1 = C_1 e^{3t} + C_2 e^{-t}, \quad y_2 = C_1 e^{3t} - C_2 e^{-t}$$

is the general solution of the homogeneous system. Let us find the particular solution of the nonhomogeneous system by Lagrange's method. We shall regard $C_1 = C_1(t)$, $C_2 = C_2(t)$ as functions of t. Then

$$\dot{y}_1 = 3C_1(t)\, e^{3t} - C_2(t)\, e^{-t} + C_1'(t)\, e^{3t} + C_2'(t)\, e^{-t},$$
$$\dot{y}_2 = 3C_1(t)\, e^{3t} + C_2(t)\, e^{-t} + C_1'(t)\, e^{3t} - C_2'(t)\, e^{-t}.$$

Substituting both these values of the derivatives and the functions themselves into our system, we obtain

$$C_1'(t)\, e^{3t} - C_2'(t)\, e^{-t} = e^t,$$
$$C_1'(t)\, e^{3t} - C_2'(t)\, e^{-t} = 0.$$

The determinant of the given system is the Wronskian

$$W = \begin{vmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{vmatrix} = -2e^{2t} \neq 0.$$

Therefore the system is solvable:

$$C_1'(t) = \frac{1}{2} e^{-2t}, \quad C_2'(t) = \frac{1}{2} e^{2t}.$$

Integrating, we get

$$C_1(t) = -\frac{1}{4}e^{-2t}, \quad C_2(t) = \frac{1}{4}e^{2t}.$$

Thus, the particular solution has the form

$$y_1(t) = 0, \quad y_2(t) = -\frac{1}{2}e^t.$$

The general solution can be written in the form

$$y_1(t) = C_1e^{3t} + C_2e^{-t}, \quad y_2(t) = C_1e^{3t} - C_2e^{-t} - \frac{1}{2}e^t.$$

Find this in the vector (matrix) form:

$$\mathbf{y} = C_1\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} + C_2\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2}e^t \end{pmatrix}.$$

Below we shall give a couple of examples to show how the particular solution of system (1) can be found when $f_i(t) = b_i \exp(\lambda_i t)$ $(i = 1, \ldots, n)$, where b_i, λ_i are given constant numbers.

If the right-hand sides of $f_i(t)$ have a special form ($\exp(kt)$ or $t^m \exp(kt)$), then the particular solution of a nonhomogeneous linear system of equations with constant coefficients can be found by analogy with the solution of a nonhomogeneous differential equation.

Example 2. Solve the system

$$\frac{dx}{dt} = x + 2y + e^t,$$

$$\frac{dy}{dt} = 2x + y + e^{3t}.$$

We first solve the homogeneous system. The characteristic equation

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

has the roots $\lambda_1 = 3$, $\lambda_2 = -1$. Hence, the general solution of the homogeneous system will be written in the form

$$x = C_1e^{3t} + C_2e^{-t}, \quad y = C_1e^{3t} - C_2e^{-t}.$$

To the free terms of the system $f_1(t) = e^t$, $f_2(t) = e^{3t}$ there correspond the numbers 1 and 3. The number 1 is not a root of the characteristic equation, while 3 is its root of the second multiplicity. By analogy, as for one equation, we set

$$\bar{x} = b_1 e^t + (b_2 + b_3 t) e^{3t},$$
$$\bar{y} = a_1 e^t + (a_2 + a_3 t) e^{3t}.$$

Substituting these functions into our system, we find

$$b_1 = 0, \quad b_2 = 0, \quad b_3 = 1/2;$$
$$a_1 = -1/2, \quad a_2 = 1/4, \quad a_3 = 1/2.$$

The general solution of the nonhomogeneous system will be written as follows:

$$x = C_1 e^{3t} + C_2 e^{-t} + \frac{t}{2} e^{3t},$$
$$y = C_1 e^{3t} - C_2 e^{-t} - \frac{1}{2} e^t + \left(\frac{1}{4} + \frac{t}{2}\right) e^{3t}.$$

Sec. 1.24. Integration of Differential Equations with the Aid of Power Series

The present section considers two examples of solution of second-order linear differential equations with the aid of power series, whose coefficients are some polynomials.

The second example deals with Bessel's differential equation which is of importance in mathematics and its applications. The solutions of Bessel's equation, constituting its fundamental system of functions, are not elementary functions. But they, as we shall see, are expanded in power series whose coefficients are computed in a rather simple way.

Example 1. $y'' - xy = 0$, $y(0) = 0$, $y'(0) = 1$.

In this case $p_0(x) = -x$, i.e. it is a polynomial of degree 1 with respect to x. We shall seek for a solution in the form of the series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{1}$$

By virtue of the initial condition $y(0) = 0$, we obtain that $a_0 = 0$. From the condition $y'(0) = 1$ we get $a_1 = 1$. Per-

forming formally termwise differentiation twice and substituting into the equation, we obtain

$$\sum_{k=2}^{\infty} k(k-1)\,a_k x^{k-2} = x \sum_{k=0}^{\infty} a_k x^k. \tag{2}$$

Comparing the coefficients of like powers of x in both members of (2), we obtain

$$a_2 = 0;\quad 3\cdot 2a_3 = a_0,\ \text{whence}\quad a_3 = \frac{a_0}{2\cdot 3} = 0;$$

$$4\cdot 3a_4 = a_1,\ \text{whence}\quad a_4 = \frac{a_1}{3\cdot 4};$$

$$5\cdot 4a_5 = a_2,\ \text{whence}\quad a_5 = 0;$$

$$6\cdot 5a_6 = a_3,\ \text{whence}\quad a_6 = \frac{a_3}{5\cdot 6} = 0;$$

. . . . . .

$$n(n-1)\,a_n = a_{n-3},\ \text{whence}\ a_n = \frac{a_{n-3}}{n(n-1)};$$

. . . . . .

Since $a_0 = a_2 = 0$, all the coefficients

$$a_{3k} = 0,\quad a_{3k+2} = 0.$$

Further

$$a_{3k+1} = \frac{a_{3(k-1)+1}}{(3k+1)\,3k} = \ldots = \frac{1}{3\cdot 4\cdot 6\cdot 7 \ldots 3k\,(3k+1)}$$

$$(k = 1, 2, \ldots).$$

Thus,

$$y(x) = x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \ldots + \frac{x^{3k+1}}{3\cdot 4\cdot 6\cdot 7 \ldots 3k\,(3k+1)} + \ldots. \tag{3}$$

According to D'Alembert's test, the radius of convergence of this series is equal to infinity. Consequently, all our operations were legitimate and the sum of the series is a solution of the equation for all values of x.

Example 2. *Bessel's* equation*:

$$x^2y'' + xy' + (x^2 - \nu^2)\,y = 0. \tag{4}$$

* Bessel, Friedrich Wilhelm (1784-1846). German astronomer and mathematician.

Many problems of mathematical physics are reduced to this equation.

We shall seek for the particular solution of (4) in the form of the generalized power series

$$y = x^{\rho} \sum_{k=0}^{\infty} a_k x^k. \tag{5}$$

Differentiating (5) term-by-term twice and substituting into (4), we obtain

$$\sum_{k=0}^{\infty} a_k (k+\rho)^2 x^{k+\rho} + (x^2 - \nu^2) \sum_{k=0}^{\infty} a_k x^{k+\rho} = 0,$$

$$\sum_{k=0}^{\infty} a_k x^{k+\rho} [(k+\rho)^2 + x^2 - \nu^2] = 0.$$

Comparing the coefficients of like powers of x, we obtain

$$\left.\begin{array}{ll} x^{\rho} & a_0 (\rho^2 - \nu^2) = 0, \\ x^{\rho+1} & a_1 [(\rho+1)^2 - \nu^2] = 0, \\ x^{\rho+2} & a_2 [(\rho+2)^2 - \nu^2] + a_0 = 0, \\ x^{\rho+3} & a_3 [(\rho+3)^2 - \nu^2] + a_1 = 0, \\ \cdots & \cdots\cdots\cdots\cdots\cdots \\ x^{\rho+p} & a_p [(\rho+p)^2 - \nu^2] + a_{p-2} = 0, \\ \cdots & \cdots\cdots\cdots\cdots\cdots \end{array}\right\} \tag{6}$$

The coefficient a_0 of the lowest power of x will be regarded as different from zero; then the first equation in (6) yields

$$\rho = \pm\nu.$$

Let for the time being $\rho = \nu \geqslant 0$. Then from the second equation of (6) we obtain $a_1 [(\nu+1)^2 - \nu^2] = 0$, i.e. $a_1 = 0$, and consequently, all the coefficients with odd numbers are also equal to zero ($a_{2p+1} = 0$). Further

$$\left.\begin{array}{l} a_2 = -\dfrac{a_0}{(\nu+2)^2 - \nu^2} = -\dfrac{a_0}{2^2 (\nu+1)}, \\ a_4 = -\dfrac{a_2}{(\nu+4)^2 - \nu^2} = \dfrac{a_0}{2^4 (\nu+1)(\nu+2) 1 \cdot 2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{2p} = \dfrac{(-1)^p a_0}{2^{2p} p! (\nu+1)(\nu+2)\ldots(\nu+p)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \tag{7}$$

For $\rho = -\nu$ we obtain in the same way

$$a_{2p+1} = 0, \quad a_{2p} = \frac{(-1)^p a_0}{2^{2p} p! (-\nu+1)(-\nu+2)\ldots(-\nu+p)}.$$

Thus, for $\rho = \nu$ the solution of equation (4) will be written as follows:

$$y = a_0 \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p+\nu}}{2^{2p} p! (\nu+1)\ldots(\nu+p)}. \tag{8}$$

Consider the function

$$\Gamma(p) = \int_0^{\infty} e^{-x} x^{p-1}\, dx \quad (p > 0),$$

called Euler's gamma function. It is readily checked that $\Gamma(p+1) = p\Gamma(p)$ and that for integral $p > 0$, $\Gamma(p+1) = p!$.

For negative p's $\Gamma(p)$ is determined in a different way, but the property $\Gamma(p+1) = p\Gamma(p)$ remains valid.

If we choose an arbitrary constant $a_0 = 1/(2^{\nu}\Gamma(\nu+1))$, then (8) will be written as

$$J_{\nu}(x) = y = \sum_{p=0}^{\infty} \frac{(-1)^p (x/2)^{2p+\nu}}{\Gamma(p+1)\Gamma(p+\nu+1)}. \tag{9}$$

For $\rho = -\nu$, choosing $a_0 = 1/(2^{-\nu}\Gamma(-\nu+1))$, we obtain analogously

$$J_{-\nu}(x) = y = \sum_{p=0}^{\infty} \frac{(-1)^p (x/2)^{2p-\nu}}{\Gamma(p+1)\Gamma(p-\nu+1)}. \tag{10}$$

The functions $J_{\nu}(x)$ and $J_{-\nu}(x)$ are called the *modified Bessel functions of the first kind of order* ν *and* $-\nu$, respectively.

Series (9) converges for all x's, and series (10) $\forall x \neq 0$; both of them allow repeated termwise differentiation. Consequently, $J_{\nu}(x)$ and $J_{-\nu}(x)$ are solutions of Bessel equation (4).

If ν is not an integer, then the functions $J_{\nu}(x)$ and $J_{-\nu}(x)$ are linearly independent, since their series begin with different powers of x and the linear combination

$$\alpha_1 J_{\nu}(x) + \alpha_2 J_{-\nu}(x) \equiv 0$$

only for $\alpha_1 = \alpha_2 = 0$. Therefore in this case the general solution of (4) has the form

$$y = C_1 J_\nu(x) + C_2 J_{-\nu}(x).$$

If $\nu = n$ is an integer, then it can be shown that

$$J_{-n}(x) = (-1)^n J_n(x),$$

i.e. these functions turn out to be linearly dependent ($\Gamma(p)$ for negative integers p and $p = 0$ turns into infinity). Therefore some other function should be taken for a second

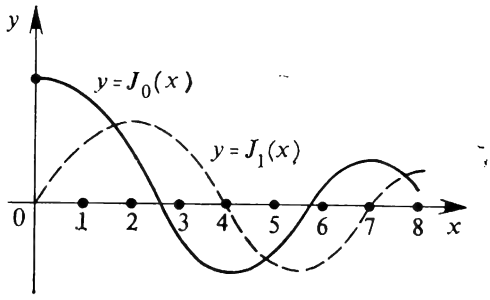

Fig. 16.

linearly independent solution. Usually, Bessel's function of the second kind $Y_n(x)$ is taken. This function is a certain combination of the functions $J_m(x)$ and $J_{-m}(x)$:

$$Y_n(x) = \lim_{\nu \to n} \frac{J_\nu(x)\cos \nu\pi - J_{-\nu}(x)}{\sin \nu\pi}$$

(ν is not an integer and tends to n).

Remark 1. Thus, the general solution of equation (4) for natural $\nu = n$ has the form

$$y = C_1 J_n(x) + C_2 Y_n(x),$$

where C_1, C_2 are arbitrary constants. Note that the function $Y_n(x)$ is not bounded in the neighbourhood of $x = 0$. For instance,

$$\pi Y_0(x) = 2J_0(x)\left[\ln \frac{x}{2} + C\right] - 2\sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} \sum_{m=1}^{k} \frac{1}{m},$$

where C is Euler's constant ($C \approx 0.577$).

Therefore any solution of equation (4), bounded in the neighbourhood of $x = 0$ has the form $y = C_1 J_n(x)$, that is, for it $C_2 = 0$.

Figure 16 represents the graphs of an even ($J_0(x)$) and odd ($J_1(x)$) Bessel's functions.

Sec. 1.25. Elements of Stability Theory

In solving many problems it is significant to know not only one concrete solution of a given problem responding to given initial conditions, but also the behavior of the solution when the initial solutions and the argument vary. These questions are answered by the qualitative theory of differential equations one of whose aspects is the *solution stability theory* or dynamic stability theory.

Let a certain phenomenon be described by a system of differential equations

$$\frac{dy_i}{dt} = f_i(t, y_1, \ldots, y_n) \quad (i = 1, \ldots, n) \tag{1}$$

with the initial conditions

$$y_i(t_0) = y_{i0} \quad (i = 1, \ldots, n). \tag{2}$$

Conditions (2) are usually the result of measurement and, consequently, are obtained with a certain accuracy.

If arbitrarily small variations of the initial data are able to change greatly the solution, then the solution of system (1) defined by the chosen inexact initial data is of no significance and cannot describe a certain phenomenon even approximately.

Therefore it is important to know the conditions under which a small change in conditions (2) involves a small change of the solution of system (1).

If t varies in a sufficiently small finite interval $|t_0 - t| \leqslant T$, then the answer to this question can be obtained on the basis of the existence and uniqueness theorem.

Theorem 1 (on continuous dependence of a solution on the initial conditions). *If the right-hand member of the differential equation*

$$\frac{dy}{dt} = f(t, y) \tag{3}$$

is continuous and has a bounded partial derivative with respect to y ($|f'_y| \leqslant N$) *in the rectangle* $D = \{t_0 - a \leqslant t \leqslant t_0 + a, y_0 - b \leqslant y \leqslant y_0 + b\}$, *then the solution of equation* (3) $y(t) = y(t, t_0, y_0)$, *satisfying the initial condition* $y(t_0) = y_0$ *depends continuously on the initial data. A more precise wording: for any* $\varepsilon > 0$ *there is a number* $\delta > 0$ *such that if* $|y_0 - \bar{y}_0| < \delta$, *then*

$$|y(t, t_0, y_0) - y(t, t_0, \bar{y}_0)| < \varepsilon$$

for

$$|t_0 - t| < T, \quad T < T_0, \quad T_0 = \min\left\{a, \frac{1}{N}, \frac{b}{M}\right\},$$

$$M = \max_{(t, y) \in D} |f(t, y)|.$$

Proof. When proving the existence theorem (Sec. 1.6) we obtained that

$$y(t) = y(t, t_0, y_0) = y_0 + \int_{t_0}^{t} f(t, y(t))\, dt,$$

$$\bar{y}(t) = y(t, t_0, \bar{y}_0) = \bar{y}_0 + \int_{t_0}^{t} f(t, \bar{y}(t))\, dt.$$

Hence, applying Lagrange's theorem (see the explanations below), we obtain

$$|y(t) - \bar{y}(t)| \leqslant |y_0 - \bar{y}_0| + \int_{t_0}^{t} |f(t, y(t)) - f(t, \bar{y}(t))|\, dt \leqslant |y_0 - \bar{y}_0| + N|t - t_0| \max_t |y(t) - \bar{y}(t)|$$

$$\leqslant |y_0 - \bar{y}_0| + NT \max_t |y(t) - \bar{y}(t)|.$$

Since $TN < 1$, we have

$$(1 - TN)\, |y(t) - \bar{y}(t)| \leqslant |y_0 - \bar{y}_0|,$$

or

$$|y(t) - \bar{y}(t)| \leqslant |y_0 - \bar{y}_0|/(1 - TN).$$

If we now take

$$|y_0 - \bar{y}_0| < \delta, \text{ where } \delta = \varepsilon(1 - TN),$$

then

$$| y(t) - \bar{y}(t) | < \varepsilon.$$

Let us give some explanations. There takes place the inequality

$$| y(t, t_0, y_0) - y_0 | \leqslant \left| \int_{t_0}^{t} f(t, y(t))\, dt \right| \leqslant | t - t_0 | M \leqslant TM,$$

where $TM \leqslant T_0 M < b$ or $b - TM = \delta_0 > 0$, showing that the integral curve $y(t, t_0, y_0)$ for $| t - t_0 | < T$ belongs to a rectangle found strictly inside the rectangle D. Hence, it is evident that the curve

$$y(t, t_0, \bar{y}_0), \quad | t - t_0 | < T,$$

also does not exceed the bounds of the rectangle D if only

$$| \bar{y}_0 - y_0 | < \delta_0.$$

This shows that the above application of Lagrange's theorem was justified, in any case for $\delta < \delta_0$, since, by the hypothesis, the function $f(t, y)$ has a continuous derivative $\frac{\partial f}{\partial y}$ in D.

A similar theorem holds true for system (1) as well.

If all the conditions of the theorem are fulfilled, we say that a problem is set correctly.

Hitherto we have studied the solution stability within a sufficiently small interval of variation of t.

But if the argument $t \in [t_0, \infty)$ attains any values, then the question of dependence of the solution on the initial data is answered by the stability theory.

Definition. Let $\varphi(t) = \{\varphi_1(t), \ldots, \varphi_n(t)\}$ be a solution of system (1). This solution is said to be a *Lyapunov* stable solution* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution $\mathbf{y}(t) = \{y_1(t), \ldots, y_n(t)\}$ of the same system whose initial values satisfy the inequalities

$$| y_i(t_0) - \varphi_i(t_0) | < \delta \quad (i = 1, \ldots, n), \qquad (4)$$

* Lyapunov, Aleksandr Mikhailovich (1857-1918). Outstanding Russian mathematician and mechanical engineer.

the following inequalities hold:

$$|y_i(t) - \varphi_i(t)| < \varepsilon \quad (i = 1, \ldots, n), \ \forall\, t \in [t_0, \infty). \tag{5}$$

Thus, the solution $\varphi(t)$ is Lyapunov stable if the solutions close to it by the initial conditions also remain close for $t \geqslant t_0$ (Fig. 17).

If a solution $\varphi(t)$ is Lyapunov stable and, in addition,

$$\lim_{t \to \infty} |y_i(t) - \varphi_i(t)| = 0 \quad (i = 1, \ldots, n), \tag{6}$$

then it is said to be *asymptotically stable.*

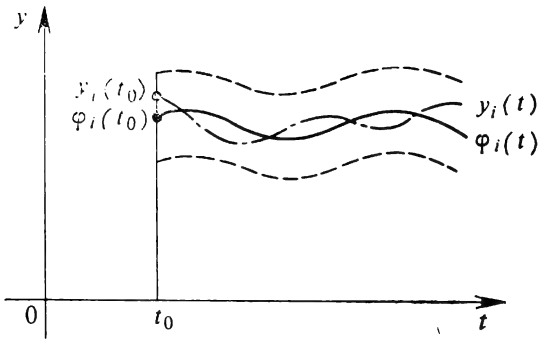

Fig. 17.

Note that (6) does not imply the Lyapunov stability.

Example 1. $\frac{dy}{dx} = -y$, $y(x_0) = y_0$.

The general solution of this equation is $y = C \exp(-x)$. The solution satisfying the initial condition has the form

$$y = y_0 \exp(x_0 - x).$$

If now we are given another initial condition $\bar{y}(x_0) = \bar{y}_0$, then the solution will be

$$\bar{y} = \bar{y}_0 \exp(x_0 - x).$$

Hence

$$|y - \bar{y}| = |y_0 - \bar{y}_0| \exp(x_0 - x) \leqslant |y_0 - \bar{y}_0|,$$

for $x \geqslant x_0$. Therefore, if $|y_0 - \bar{y}_0| < \delta = \varepsilon$, then $|y - \bar{y}| \leqslant \varepsilon$, i.e. the solution $y = y_0 \exp(x_0 - x)$ is Lyapunov stable for $x \geqslant x_0$. This solution is also asymptotically stable

since

$$\lim_{x \to +\infty} | y - \bar{y} | = \lim_{x \to +\infty} | y_0 - \bar{y}_0 | \exp (x_0 - x) = 0.$$

Example 2. Analogously, for the equation $y' = y$ we can show that

$$| y - \bar{y} | = | y_0 - \bar{y}_0 | \exp (x - x_0)$$

for $x \geqslant x_0$ for any x_0.

Obviously, whatever x_0 in $x \geqslant x_0$ may be, the solution y is unstable, since the factor $\exp (x - x_0) \to +\infty$ as $x \to +\infty$.

The investigation for Lyapunov stability of an arbitrary solution $\varphi (t) = \{\varphi_1 (t), \ldots, \varphi_n (t)\}$ of system (1) can be reduced to the investigation of a trivial (identically equal to zero) solution of some other system. With this purpose in mind, we pass to the new unknown functions

$$x_i (t) = y_i (t) - \varphi_i (t) \ (i = 1, \ldots, n). \tag{7}$$

Hence

$$\frac{dy_i}{dt} = \frac{dx_i}{dt} + \frac{d\varphi_i}{dt}.$$

Therefore system (1) will turn into the system

$$\frac{dx_i}{dt} = f_i [t, x_1 + \varphi_1 (t), \ldots, x_n + \varphi_n (t)] - f_i [t, \varphi_1 (t), \ldots, \varphi_n (t)] \quad (i = 1, \ldots, n). \tag{8}$$

System (8) has the trivial solution

$$x_i (t) \equiv 0 \quad (i = 1, \ldots, n). \tag{9}$$

The aforesaid implies the following theorem.

Theorem 2. *The solution* $\varphi (t) = \{\varphi_1 (t), \ldots, \varphi_n (t)\}$ *of system* (1) *is Lyapunov stable (asymptotically stable) if and only if the trivial solution (stationary point) of system* (8) *is Lyapunov stable (asymptotically stable).*

This solution possesses the following property: the point $(x_1 (t), \ldots, x_n (t))$ does not move as t varies, it is actually fixed (or stationary). In this case solution (9) itself and the point $(0, \ldots, 0)$ are called the *equilibrium position of system* (1) *or stationary point.*

The stability conditions as applied to the stationary point $x_1 = 0$ $(i = 1, \ldots, n)$ can be formulated as follows: the stationary point $x_i \equiv 0$ $(i = 1, \ldots, n)$ of system (8)

is Lyapunov stable if $\forall \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$ such that from the inequality

$$|x_i(t_0)| < \delta(\varepsilon) \quad (i = 1, \ldots, n)$$

it follows

$$|x_i(t)| < \varepsilon \quad (i = 1, \ldots, n),\ \forall\, t \geqslant t_0,$$

i.e. a trajectory whose initial point is found in some δ-neighbourhood of the origin does not exceed the bounds of an arbitrary ε-neighbourhood of the origin for $t \geqslant t_0$. Here we speak of rectangular neighbourhoods, but we can also pass to spherical neighbourhoods, which is especially convenient for the vector form of notation of the solution $\mathbf{x}(t) = \{x_1(t), \ldots, x_n(t)\}$:

$$\|\mathbf{x}(t_0)\| < \delta(\varepsilon) \Rightarrow \|\mathbf{x}(t)\| < \varepsilon, \quad \|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

Remark 1. An arbitrary particular solution $\mathbf{y}_0(t)$ of the nonhomogeneous linear system of differential equations

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} + \mathbf{f}(t) \tag{10}$$

(see Sec. 1.23, (1′)) is Lyapunov stable (asymptotically stable) if and only if Lyapunov stable (asymptotically stable) is the stationary point of the corresponding homogeneous system (see Theorem 2)

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{11}$$

Indeed, system (10) is a particular case of system (1), and system (11) is a particular case of system (8). Here the free terms have vanished, since the function $\mathbf{f}(t)$ depends only on t and is independent of the sought-for functions.

Theorem 3 (Lyapunov's theorem). *Let there be given a system*

$$\frac{dy_i}{dt} = f_i(t, y_1, \ldots, y_n) \quad (i = 1, \ldots, n), \tag{1}$$

having the trivial solution $y_i(t) \equiv 0$ $(i = 1, \ldots, n)$.

Let there exist a differentiable function $v(y_1, \ldots, y_n)$ *satisfying the conditions:*

(1) $v(y_1, \ldots, y_n) \geqslant 0$ *and* $v = 0$ *only for* $y_1 = \ldots = y_n = 0$, *i.e. the function* v *has a strict minimum at the origin.*

(2) *The total derivative of the function* v *along the phase trajectory (i.e. along the solution* $y_i(t)$ *of system* (1))

$$\frac{dv}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial y_i} \frac{dy_i}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial y_i} f_i(t, y_1, \ldots, y_n) \leqslant 0 \quad \text{for } t \geqslant t_0.$$

Then the stationary point $y_i \equiv 0$ $(i = 1, \ldots, n)$ *is Lyapunov stable.*

If it is additionally required that outside an arbitrarily small neighbourhood of the origin $(y_1^2 + \ldots + y_n^2 \geqslant \delta)$

$$\frac{dv}{dt} \leqslant -\beta < 0 \quad (t \geqslant t_0),$$

where β *is a constant quantity, then the stationary point* $y_i(t) \equiv 0$ $(i = 1, \ldots, n)$ *is asymptotically stable.*

The function v *is called Lyapunov's function.*

Example 1.

$$\frac{dy_1}{dt} = -y_1^5 - y_2,$$

$$\frac{dy_2}{dt} = y_1 - y_2^3.$$

It is evident that the stationary point of $y_1 = y_2 \equiv 0$ is a solution of the given system. Let us find out whether it is stable.

Consider the function $v(y_1, y_2) = y_1^2 + y_2^2$. It satisfies all the three conditions of the theorem:

(1) $v(y_1, y_2) \geqslant 0$ and $v = 0$ only for $y_1 = y_2 = 0$.

(2) Along the phase trajectory

$$\frac{dv}{dt} = \frac{\partial v}{\partial y_1} \frac{dy_1}{dt} + \frac{\partial v}{\partial y_2} \frac{dy_2}{dt} = 2y_1(-y_1^5 - y_2) + 2y_2(y_1 - y_2^3) = -2(y_1^6 + y_2^4) \leqslant 0.$$

Besides, outside the neighbourhood of the origin $(y_1^2 + y_2^2 \geqslant \delta > 0)$

$$\frac{dv}{dt} \leqslant -\beta < 0$$

(where β is the minimum of the function $2(y_1^6 + y_2^4)$ outside the circle $y_1^2 + y_2^2 = \delta$).

Hence, the solution $y_1 = y_2 \equiv 0$ is asymptotically stable.

Remark 2. Lyapunov's function is recommended to be sought in the form of the quadratic form of the arguments $y_1, \ldots, y_n$, i.e.

$$v = \sum_{i,j} a_{ij} y_i y_j.$$

The first requirement states that v must be a positive definite quadratic form. How to choose the coefficients a_{ij} in order to get a positive definite form of v is indicated in Sylvester's* theorem

$$\left(a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ldots, \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} > 0 \right).$$

Sec. 1.26. Classification of Stationary Points

We can investigate for stability the system of two first-order linear equations with constant coefficients

$$\left.\begin{aligned} \frac{dx}{dt} &= a_{11}x + a_{12}y, \\ \frac{dy}{dt} &= a_{21}x + a_{22}y \end{aligned}\right\} \qquad (1)$$

using Lyapunov's theorem. But system (1) can be investigated for stability directly, since we can solve it.

We shall assume that the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0. \qquad (2)$$

It is easily seen that $y(t) = x(t) \equiv 0$ is the solution of system (1) with zero initial conditions.

To determine the general solution, we must find the roots of the characteristic equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \Delta = 0. \qquad (3)$$

* See our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 22.

Sylvester, James Joseph (1814-1897). English mathematician.

From the condition $\Delta \neq 0$ it follows that $\lambda = 0$ is not a root of the characteristic equation (3).

1. Let the roots of the characteristic equation λ_1 and λ_2 are real and distinct. Further, let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2)$ be the eigenvectors of matrix A responding to roots λ_1 and λ_2, respectively, i.e.

$$\left.\begin{aligned}(a_{11} - \lambda_1)\,\alpha_1 + a_{12}\alpha_2 = 0,\\ a_{21}\alpha_1 + (a_{22} - \lambda_1)\,\alpha_2 = 0,\end{aligned}\right\} \quad \left.\begin{aligned}(a_{11} - \lambda_2)\,\beta_1 + a_{12}\beta_2 = 0,\\ a_{21}\beta_1 + (a_{22} - \lambda_2)\,\beta_2 = 0.\end{aligned}\right\} \tag{4}$$

Then, as we know, the general solution of system (1) has the form

$$\left.\begin{aligned}x = C_1\alpha_1 e^{\lambda_1 t} + C_2\beta_1 e^{\lambda_2 t},\\ y = C_1\alpha_2 e^{\lambda_1 t} + C_2\beta_2 e^{\lambda_2 t},\end{aligned}\right\} \tag{5}$$

where C_1 and C_2 are arbitrary constants.

If $\lambda_1 < 0$, $\lambda_2 < 0$, then it is seen from (5) that the stationary point $x = y \equiv 0$ is asymptotically stable.

Indeed, we shall assume, for instance, that $t_0 = 0$, then solution (5) passing through the point (x_0, y_0) at an instant of time t_0 is defined by the constants C_1 and C_2 which are computed from the equations

$$\begin{aligned}x_0 &= C_1\alpha_1 + C_2\beta_1,\\ y_0 &= C_1\alpha_2 + C_2\beta_2,\end{aligned}$$

where $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. But then

$$C_1 = Ax_0 + By_0, \quad C_2 = Dx_0 + Ey_0,$$

where A, B, C, D, E are some constants. Consequently,

$$\begin{aligned}|\,x\,(t)\,| &\leqslant |\,Ax_0 + By_0\,|\,|\,\alpha_1\,| + |\,Dx_0 + Ey_0\,|\,|\,\beta_1\,|,\\ |\,y\,(t)\,| &\leqslant |\,Ax_0 + By_0\,|\cdot|\,\alpha_2\,| + |\,Dx_0 + Ey_0\,|\cdot|\,\beta_2\,|,\end{aligned}$$

since $|\,e^{\lambda_1 t}\,| \leqslant 1$, $|\,e^{\lambda_2 t}\,| \leqslant 1$ for $\lambda_1 < 0$, $\lambda_2 < 0$.

Hence it follows that for any $\varepsilon > 0$ there is $\delta > 0$ such that as soon as $|\,x_0\,|$, $|\,y_0\,| < \delta$, the inequalities

$$|\,x\,(t)\,|, \quad |\,y\,(t)\,| < \varepsilon \quad (t > 0),$$

are fulfilled, i.e. the point (0, 0) is Lyapunov stable. Besides, by virtue of the fact that

$$\exp(\lambda_i t) \to 0 \quad (t \to +\infty),$$

it follows, obviously, from (5) that the point (0, 0) is asymptotically stable.

If we eliminate the argument t from system (5), then the obtained function $y = \varphi(x)$ yields a mechanical trajectory in the xOy system.

A material point found at the initial instant $t = t_0$ in the δ-neighbourhood of the origin, at a sufficiently large t

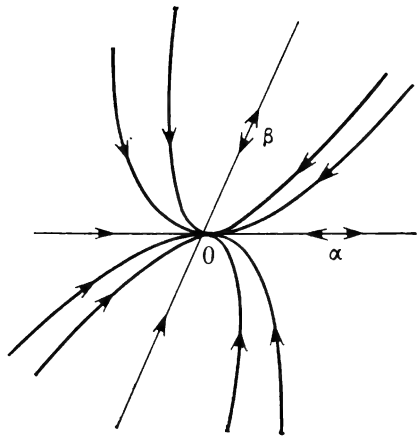

Fig. 18.

goes into a point lying in the ε-neighbourhood of the origin and tends to the origin as $t \to +\infty$.

Such stationary point is called a *stable node* (Fig. 18).

Figure 18 represents the arrangement of the trajectories corresponding to the case under consideration. The arrows indicate the direction of a point moving in its trajectory as $t \to +\infty$. All the trajectories, but one, have a common tangent at the point (0, 0). If we assume that $|\lambda_1| < |\lambda_2|$, then the slope of the tangent is equal to α_2/α_1.

Indeed, from (5) we have

$$\frac{dy}{dx} = \frac{a_{21}(C_1\alpha_1 e^{\lambda_1 t} + C_2\beta_1 e^{\lambda_2 t}) + a_{22}(C_1\alpha_2 e^{\lambda_1 t} + C_2\beta_2 e^{\lambda_2 t})}{a_{11}(C_1\alpha_1 e^{\lambda_1 t} + C_2\beta_1 e^{\lambda_2 t}) + a_{12}(C_1\alpha_2 e^{\lambda_1 t} + C_2\beta_2 e^{\lambda_2 t})}$$

$$= \frac{a_{21}(C_1\alpha_1 + C_2\beta_1 e^{(\lambda_2-\lambda_1)t}) + a_{22}(C_1\alpha_2 + C_2\beta_2 e^{(\lambda_2-\lambda_1)t})}{a_{11}(C_1\alpha_1 + C_2\beta_1 e^{(\lambda_2-\lambda_1)t}) + a_{12}(C_1\alpha_2 + C_2\beta_2 e^{(\lambda_2-\lambda_1)t})}$$

$$\xrightarrow[t\to+\infty]{} \frac{a_{21}C_1\alpha_1 + a_{22}C_1\alpha_2}{a_{11}C_1\alpha_1 + a_{12}C_1\alpha_2} = \frac{a_{21}\alpha_1 + a_{22}\alpha_2}{a_{11}\alpha_1 + a_{12}\alpha_2} = \frac{\lambda_1\alpha_2}{\lambda_1\alpha_1} = \frac{\alpha_2}{\alpha_1},$$

if $\alpha_1 \neq 0$, since by virtue of (4),

$$a_{21}\alpha_1 + a_{22}\alpha_2 = \lambda_1\alpha_2, \quad a_{11}\alpha_1 + a_{12}\alpha_2 = \lambda_1\alpha_1.$$

And if $\alpha_1 = 0$, then, reasoning in the same way, we obtain $\frac{dx}{dy} \to \frac{\alpha_1}{\alpha_2} = 0$ ($\lambda_1 \neq 0$!).

If $C_1 = 0$, then (5) yields one trajectory

$$y = \frac{\beta_2}{\beta_1} x.$$

The tangent to this trajectory (which is a straight line) has the slope β_2/β_1.

Thus, the tangent to the trajectories for which $C_1 \neq 0$ is parallel to the eigenvector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, responding to the least (by modulus) eigenvalue λ_1 (for $\alpha_1 = 0$ this vector is directed along the y-axis).

Besides, there is a trajectory (for $C_1 = 0$), namely, the straight line $y = \frac{\beta_2}{\beta_1} x$, which is parallel to the second eigenvector $\boldsymbol{\beta} = (\beta_1, \beta_2)$ responding to the greater (by modulus) eigenvalue λ_2.

If now $\lambda_1 > 0$ and $\lambda_2 > 0$, then, as is seen from (5), the stationary point $x = y \equiv 0$ is unstable, since $\exp(\lambda_i t) \to$ $\to +\infty$ as $t \to +\infty$. Such stationary point is called an *unstable node*. The present case is obtained from the preceding one by substituting $(-t)$ for t. Therefore, the trajectories will have the previous shape, but the points will move in their trajectories in a reversed sense (Fig. 19).

Finally, if the eigenvalue $\lambda_1 < 0$, and $\lambda_2 > 0$ (or, vice versa, $\lambda_1 > 0$, $\lambda_2 < 0$), then the stationary point is also unstable, since $\exp(\lambda_2 t) \to +\infty$ as $t \to +\infty$. The points found in the δ-neighbourhood of the origin go to infinity along the trajectory

$$x = C_2\beta_1 e^{\lambda_2 t}, \quad y = C_2\beta_2 e^{\lambda_2 t}.$$

Note that in the present case there is a trajectory in which the point moves towards the origin as $t \to +\infty$:

$$x = C_1\alpha_1 e^{\lambda_1 t}, \quad y = C_1\alpha_2 e^{\lambda_1 t}.$$

It is the straight line $\alpha_1 y - \alpha_2 x = 0$. Such stationary point is called a *saddle* (Fig. 20).

2. Let the roots λ_1 and λ_2 be complex (a_{kl} are real!)

$$\lambda_1 = p + iq, \quad \lambda_2 = p - iq \quad (q \neq 0).$$

The general solution of system (1) can be written in form (5), where the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} = \bar{\boldsymbol{\alpha}}$ will have complex coordinates.

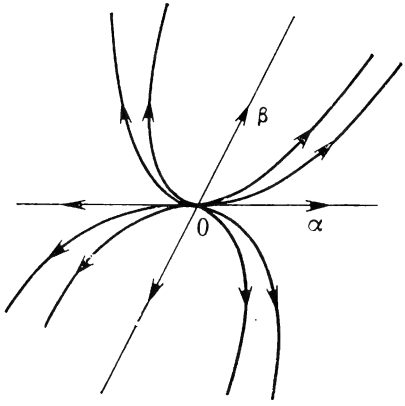

Fig. 19.

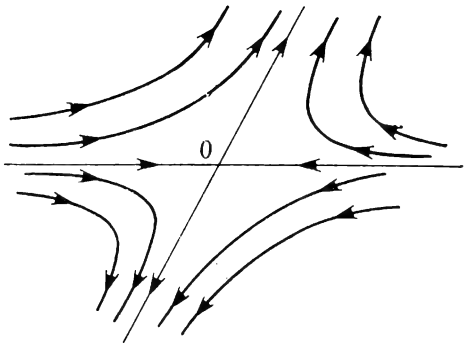

Fig. 20.

The real and the imaginary parts of this solution are also solutions of the system. Therefore the general solution of system (1) can be written as a linear combination of these solutions:

$$\left.\begin{aligned} x &= e^{pt}\,(C_1 \cos qt + C_2 \sin qt), \\ y &= e^{pt}\,(a \cos qt + b \sin qt), \end{aligned}\right\} \tag{6}$$

where C_1 and C_2 are arbitrary constants and a, b are some linear combinations of these constants

$$(a = kC_1 + lC_2, \quad b = mC_1 + nC_2).$$

Let us illustrate this fact by a particular example.

Example 1. Find the general solution of the system

$$\begin{cases} \dot{x} = x - y, \\ \dot{y} = 2x - y. \end{cases}$$

The characteristic equation

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 + 1 = 0$$

has the roots $\lambda_1 = i$, $\lambda_2 = -i$. The coordinates of the vectors α and β are found from the equalities

$$(1 - i)\,\alpha_1 - \alpha_2 = 0, \quad (1 + i)\,\beta_1 - \beta_2 = 0,$$

i.e. we may set $\alpha_1 = 1$, $\alpha_2 = 1 - i$; $\beta_1 = 1$, $\beta_2 = 1 + i$. Then

$$x = \alpha_1 e^{it} = \cos t + i \sin t,$$

$$y = \alpha_2 e^{it} = (1 - i)(\cos t + i \sin t)$$

$$= (\cos t + \sin t) + i (\sin t - \cos t)$$

is a solution of the system.

The real and imaginary parts of this solution are also solutions of the system; moreover, they are linearly independent. Therefore, their linear combination yields the general solution of our system:

$$x = C_1 \cos t + C_2 \sin t,$$

$$y = C_1 (\cos t + \sin t) + C_2 (\sin t - \cos t)$$

$$= (C_1 - C_2) \cos t + (C_1 + C_2) \sin t.$$

Thus, in this case

$$a = C_1 - C_2, \quad b = C_1 + C_2.$$

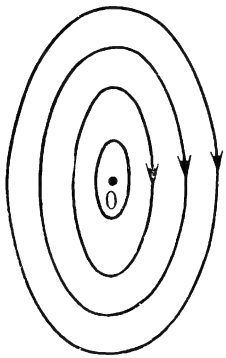

Fig. 21.

For $p = 0$ trajectories (6) for distinct C_1, C_2, by virtue of the periodicity of the factors in the parentheses, are closed curves, viz. ellipses with centre at the point (0, 0) (Fig. 21). This point is called the centre. For $p = 0$ there is no asymptotic stability—the point $(x\,(t),\ y\,(t))$ moves along one of the ellipses, going around it an infinite number of times. It does not tend to any limit as $t \to +\infty$. On the other hand, for $p = 0$ the stationary point (0, 0) is Lyapunov stable. Let us verify this assertion using the above considered Example 1. Let us find the solution of the system given in this example passing through the point x_0, y_0

at the instant of time $t_0 = 0$. Obviously,

$$x_0 = C_1, \quad y_0 = C_1 - C_2,$$

consequently,

$$C_1 = x_0, \quad C_2 = x_0 - y_0$$

and the solution has the form

$$x(t) = x_0 \cos t + (x_0 - y_0) \sin t,$$
$$y(t) = y_0 \cos t + (2x_0 - y_0) \sin t.$$

We further have

$$|x(t)| \leqslant |x_0| + |x_0| + |y_0| = 2|x_0| + |y_0|,$$
$$|y(t)| \leqslant |y_0| + 2|x_0| + |y_0| = 2|x_0| + 2|y_0|.$$

Let us take $\varepsilon > 0$, and let $\delta = \varepsilon/4$. Obviously, if $|x_0|$, $|y_0| < \delta$, then for all t's

$$|x(t)| \leqslant 2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon,$$

$$|y(t)| \leqslant 2 \cdot \frac{\varepsilon}{4} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

Thus, we have proved the Lyapunov stability of the stationary point.

It is seen from (6) that for $p < 0$ the point (x, y), as $t \to +\infty$, tends to the zero point $x = 0$, $y = 0$ called

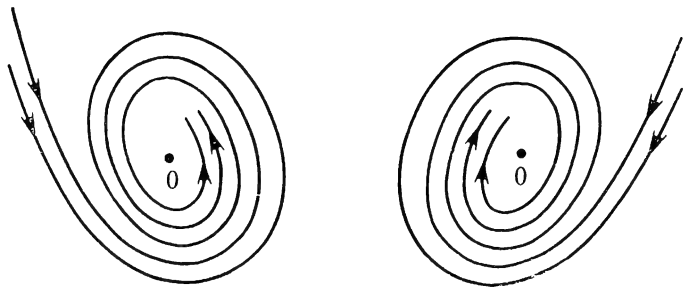

Fig. 22.

a *stable focus*. The presence of the factor $\exp(pt) \to 0$ $(t \to +\infty)$ transforms the closed curves into spirals approaching asymptotically the origin as $t \to +\infty$ (Fig. 22).

The points found for $t = t_0$ in an arbitrary δ-neighbourhood of the origin for a sufficiently large t enter into a given ε-neighbourhood of the point (0, 0).

The trajectories tending to the focus possess the following property: the tangents to them do not tend to any limit as $t \to +\infty$, a feature by which a focus is distinguished from a node.

If $p < 0$ the point $x = y = 0$ is then asymptotically stable.

If the real part p of the roots λ_1 and λ_2 is positive, then this case is reduced to the preceding one by replacing

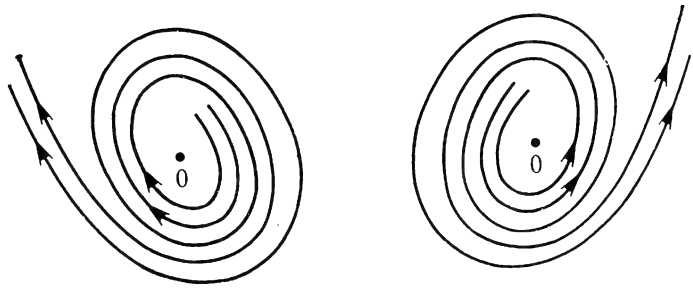

Fig. 23.

t by $(-t)$. Consequently, the trajectories retain their shape (as in Fig. 22), only the point will move in an opposite direction. Since $\exp(pt) \to +\infty$ as $t \to +\infty$, the points found in the neighbourhood of the origin at the initial instant of time then go to infinity. Such stationary point is termed an *unstable focus* (Fig. 23).

3. Let the roots λ_1, λ_2 be equal to each other ($\lambda_1 = \lambda_2$, a_{kl} is real!). Then they are real and the general solution of system (1) has the form

$$\left.\begin{aligned} x &= (A + Bt)\, e^{\lambda_1 t}, \\ y &= (C + Dt)\, e^{\lambda_1 t}, \end{aligned}\right\} \tag{7}$$

where A, B, C, D are constants related by two linear equations, which can be obtained by substituting the functions x and y into the system and reducing by the factor $\exp(\lambda_1 t)$.

If $\lambda_1 < 0$, then $\exp(\lambda_1 t) \to 0$, $t \exp(\lambda_1 t) \to 0$ as $t \to +\infty$, and, consequently, the stationary point $x = 0$, $y = 0$ is asymptotically stable. It is called a stable node (as in item 1).

But if $\lambda_1 > 0$, then the point $x = 0$, $y = 0$ is unstable and it is called an unstable node.

Remark 1. If $\Delta = 0$, then the characteristic equation (3) has the root $\lambda_1 = 0$ and $\lambda_2 = a_{11} + a_{22}$.

Let $\lambda_2 \neq 0$. Then the general solution of system (1) will be written in the following way:

$$x = C_1\alpha_1 + C_2\beta_1 e^{\lambda_2 t},$$
$$y = C_1\alpha_2 + C_2\beta_2 e^{\lambda_2 t},$$

where C_1, C_2 are arbitrary constants and $a_{11}\alpha_1 + a_{12}\alpha_2 = 0$, $-a_{22}\beta_1 + a_{12}\beta_2 = 0$.

Eliminating the parameter t, we obtain a family of parallel straight lines

$$y - C_1\alpha_2 = \frac{\beta_2}{\beta_1}(x - C_1\alpha_1).$$

If $\lambda_2 < 0$, then on each trajectory (on one of the parallel rays) as $t \to +\infty$, the points come nearer to the stationary point $x = C_1\alpha_1$, $y = C_1\alpha_2$ $\left(y = \frac{\alpha_2}{\alpha_1}x\right)$ lying on this trajectory (Fig. 24).

The point (0, 0), like any point of the straight line $y = \frac{\alpha_2}{\alpha_1}x$, is Lyapunov stable for $\lambda_2 < 0$, but is not asymptotically stable.

If $\lambda_2 > 0$, then the stationary point is unstable.

And if $\lambda_1 = \lambda_2 = 0$, then two cases are possible:

(a) The general solution of system (1) has the form $x = C_1$, $y = C_2$. In this case the stationary point $x = y \equiv 0$ is Lyapunov stable, but is not asymptotically stable. Note that this takes place when matrix A is zero: $a_{11} = a_{22} = a_{12} = a_{21} = 0$. In the present case all the points of the xy-plane are Lyapunov stable stationary points.

(b) The general solution of system (1) has the form

$$x = C_1 + C_2 t, \quad y = a + bt.$$

The stationary point $x = y \equiv 0$ is unstable.

In this case $a_{22} = -a_{11}$, $a_{12} \cdot a_{21} \leqslant 0$.

Example 2. Find out the character of the stationary point of the system

$$\dot{x} = -x + ay,$$

$$\dot{y} = -2y.$$

We set up the characteristic equation:

$$\begin{vmatrix} -1-\lambda & a \\ 0 & -2-\lambda \end{vmatrix} = 0.$$

Its roots are: $\lambda_1 = -1$, $\lambda_2 = -2$. Hence, the stationary point $x = y \equiv 0$ is a stable node.

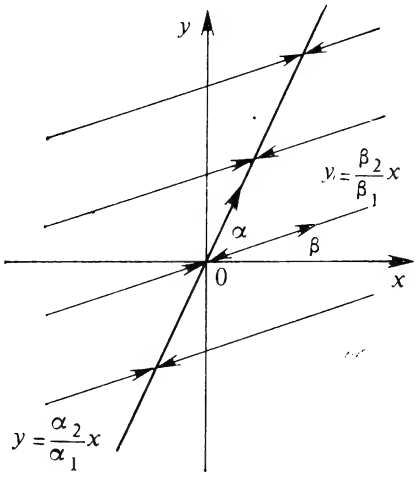

Fig. 24.

Example 3. Find out the type of the stationary point of the system

$$\dot{x} = x - y,$$

$$\dot{y} = 2x + 3y.$$

The characteristic equation

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

has the complex roots $\lambda_1 = 2 + i$, $\lambda_2 = 2 - i$. The real part of these conjugate roots is positive, therefore the stationary point $x = y \equiv 0$ is an unstable focus.

Remark 2. If matrix A is symmetric, then, as we know, the characteristic equation has only real roots*. Besides, we know that

$$a_{11}a_{22} - a_{12}^2 = \lambda_1 \cdot \lambda_2.$$

Since a symmetric matrix generates a quadratic form of the elliptic, hyperbolic, or parabolic type, the system of differential equations will be called in this case

elliptic if $a_{11}a_{22} - a_{12}^2 = \lambda_1\lambda_2 > 0$,
hyperbolic if $a_{11}a_{22} - a_{12}^2 = \lambda_1\lambda_2 < 0$,
parabolic if $a_{11}a_{22} - a_{12}^2 = \lambda_1\lambda_2 = 0$.

From the aforegoing it is clear that if system (1) is an elliptic one, then the stationary point will be a stable node, provided $\lambda_1 < 0$, $\lambda_2 < 0$. This is possible when $a_{11} < 0$, $a_{22} < 0$.

But if $\lambda_1 > 0$, $\lambda_2 > 0$, then the stationary point is an unstable node ($a_{11} > 0$, $a_{22} > 0$).

Note that in the elliptic case the numbers a_{11} and a_{22} are of the same sign.

If system (1) is hyperbolic ($a_{11}a_{22} - a_{12}^2 < 0$), then the stationary point is always unstable (a saddle point).

If system (1) is parabolic, then the stationary point is stable when $\lambda_2 = a_{11} + a_{22} \leqslant 0$, and unstable when $\lambda_2 = a_{11} + a_{22} > 0$.

Example 4. Determine the character of the stationary point of the following systems:

(a) $\dot{x} = -3x + 2y$, $\dot{y} = 2x - 5y$; (b) $\dot{x} = x + 2y$, $\dot{y} = 2x + 3y$; (c) $\dot{x} = x + \sqrt{3}\,y$, $\dot{y} = \sqrt{3}\,x + 3y$.

In all the given systems matrix A is symmetric.

System (a) is elliptic, since $a_{11}a_{22} - a_{12}^2 = 11 > 0$. Since $a_{11} = -3 < 0$, $a_{22} = -5 < 0$, then the stationary point is a stable node.

System (b) is hyperbolic, since $a_{11}a_{22} - a_{12}^2 = -1 < 0$. Hence, the stationary point is a saddle.

* See our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 24.

System (c) is parabolic, since $a_{11}a_{22} - a_{12}^2 = 0$. Since $a_{11} + a_{22} = 4 > 0$, the stationary point is unstable.

Remark 3. We can prove (just in the same manner as it was done for $n = 2$) that the stationary point is automatically Lyapunov stable (asymptotically stable) for a homogeneous linear system of $n \geqslant 1$ equations with constant coefficients

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y},$$

if all the roots of the characteristic equations of the system have negative real parts.

CHAPTER 2

MULTIPLE INTEGRALS

Sec. 2.1. Introduction

Let there be given a continuous surface, lying in the three-dimensional space with rectangular coordinates (x, y, z), which is determined by an equation

$$z = f(Q) = f(x, y) \quad (Q = (x, y) \in \Omega),$$

where Ω is a bounded (two-dimensional) set for which it is possible to define the notion of its area (two-dimensional measure, see Sec. 2.2). For example, Ω can be a circle, a rectangle, an ellipse, etc. We shall suppose that the function $f(x, y)$ is positive. Let us state the following problem: it is required to find the volume of the solid bounded above by the given surface and below by the plane $z = 0$, its lateral boundary being the cylindrical surface with the generatrix parallel to the z-axis and passing through the boundary curve γ of the plane set Ω.

It is natural to determine the sought-for volume in the following way. We separate the set Ω into a finite number N of parts (subdomains)

$$\Omega_1, \ldots, \Omega_N, \tag{1}$$

any two of which either do not intersect or intersect only along some parts of their boundaries. Let these subdomains be such that they possess areas (two-dimensional measures) which we shall denote as $m\Omega_1, \ldots, m\Omega_N$, respectively.

Let us introduce the notion of the diameter of a set: if A is a set in the plane, then its diameter $d(A)$ is defined as

$$d(A) = \sup_{P', P'' \in A} |P' - P''|,$$

where the supremum is taken over all the pairs of points P', P'' belonging to A.

Now we choose an arbitrary point $Q_j = (\xi_j, \eta_j)$ $(j = 1, \ldots, N)$ in each part Ω_j and form the sum

$$V_N = \sum_{j=1}^{N} f(Q_j)\, m\Omega_j, \tag{2}$$

which can be regarded as an approximation to the sought-for volume V. We can naturally suppose that the smaller the diameters $d(\Omega_j)$ of the subdomains Ω_j are, the higher is the accuracy of the approximation $V \approx V_N$. Therefore the volume V of the solid under consideration can be defined as the limit of sum (2)

$$V = \lim_{\max d(\Omega_j) \to 0} \sum_{j=1}^{N} f(Q_j)\, m\Omega_j, \tag{3}$$

when the maximum diameter of the subdomains of partitions (1) tends to zero, provided that this limit exists and is equal to one and the same number irrespective of the way in which the sequence of partitions (1) is chosen.

Now we can abstract from the problem of finding the volume of a solid and regard expression (3) as the result of an operation performed on the given function f defined in Ω. It is called the *Riemann* double integral of the function f over domain Ω* and is denoted

$$V = \lim_{\max d(\Omega_j) \to 0} \sum_{j=1}^{N} f(Q_j)\, m\Omega_j = \iint_{\Omega} f(x, y)\, dx\, dy$$

$$= \int_{\Omega} f(Q)\, dQ = \int_{\Omega} f\, d\Omega.$$

Let us now consider a problem leading to the notion of the triple integral. Suppose that there is a physical body Ω occupying a domain (set) in the three-dimensional space with rectangular coordinates (x, y, z) and that the mass of the body is distributed nonuniformly over it with volume density $\mu(x, y, z) = \mu(Q)$ $(Q = (x, y, z) \in \Omega)$. It is required to determine the total mass of the body Ω.

* Riemann, Georg Friedrich Bernhard (1826-1866). Great German mathematician.

To solve this problem, it is natural to partition Ω into N parts $\Omega_1, \ldots, \Omega_N$ whose volumes (three-dimensional measures) are $m\Omega_1, \ldots, m\Omega_N$ (on condition that these volumes exist), to choose an arbitrary point $Q_j = (x_j, y_j, z_j) \in \in \Omega_j$ $(j = 1, \ldots, N)$ in each of the parts and to define the sought-for mass as the limit

$$M = \lim_{\max d(\Omega_j) \to 0} \sum_{j=1}^{N} \mu(Q_j)\, m\Omega_j. \tag{4}$$

Expression (4) can again be regarded as the result of an operation performed on the function μ defined in the three-dimensional set Ω. It is called the *Riemann triple integral of f on Ω* and is denoted as

$$M = \lim_{\max d(\Omega_j) \to 0} \sum_{j=1}^{N} \mu(Q_j)\, m\Omega_j = \int_{\Omega} \mu(Q)\, dQ$$

$$= \iiint_{\Omega} \mu(x, y, z)\, dx\, dy\, dz.$$

The *Riemann n-fold multiple integral* is defined in the same way.

We shall see that the theory of (Riemann) multiple integration which includes existence theorems and theorems on the additive properties of the integral can be presented for the n-dimensional case in exactly the same manner as in the one-dimensional case. However, the theory of multiple integrals involves some specific difficulties which were not encountered in the theory of one-fold integration.

The point is that the Riemann one-fold integral was defined for an extremely simple set, namely, for a closed interval $[a, b]$ which was partitioned into parts which were also closed intervals. Therefore we had no difficulties in defining the lengths (one-dimensional measure) of the intervals. But in the case of a double integral or, generally, n-fold integral, the domain of integration Ω can be split into parts with curvilinear boundaries, which makes it necessary to define the notion of the area or, generally, of the n-dimensional measure of such a part.

In the two-dimensional case we shall deal with bounded domains having a smooth boundary (Fig. 25) or a piecewise

smooth boundary (Fig. 26), i.e. consisting of a finite number of smooth pieces (lines).

These domains, in their turn, are divided into parts with a piecewise smooth boundary.

Each such domain ω and some other sets can be associated with a positive number $m\omega$ called the *area* or the *Jordan** *two-dimensional measure* (the general definition of the Jordan measure is given in the next section).

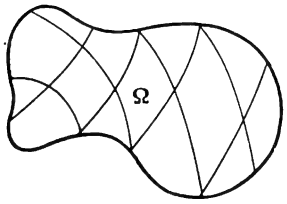

Fig. 25.

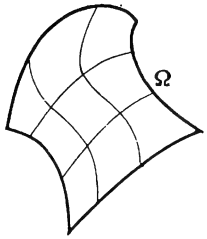

Fig. 26.

The following properties are fulfilled:

(1) If Δ is a rectangle with base a and altitude b, then $m\Delta = |\Delta| = ab$.

(2) If $\omega_1 \subset \omega_2$ and ω_1, ω_2 have measures $m\omega_1$, $m\omega_2$, respectively, then $m\omega_1 \leqslant m\omega_2$.

(3) If a domain ω is cut by a piecewise smooth curve into two parts ω_1 and ω_2 ($\omega = \omega_1 + \omega_2$), then $m\omega = m\omega_1 + m\omega_2$.

There are sets of the two-dimensional measure zero, such as a point, a line segment, a smooth or a piecewise smooth curve.

In the three-dimensional case we are interested in domains having piecewise smooth surfaces as their boundaries. Such domains will be said to have a piecewise smooth boundary. A sphere, an ellipsoid, a cube are examples of such domains.

A surface is said to be *smooth* if a tangent plane can be drawn to it at any of its points and if this tangent plane varies together with this point. A surface is called *piecewise smooth* if it can be cut into a finite number of smooth pieces. Along the division lines tangent planes to the surface in question may not exist.

* Jordan, Camille (1838-1922). French mathematician.

For three-dimensional bounded domains ω with piecewise smooth boundaries we can determine their volume (the three-dimensional measure), i.e. a positive number $m\omega$ which satisfies the following properties:

(1) If Δ is a rectangular parallelepiped with edges a, b, and c, then $m\Delta = |\Delta| = abc$.

(2) If $\omega_1 \subset \omega_2$ and ω_1, ω_2, have measures $m\omega_1$, $m\omega_2$, then $m\omega_1 \leqslant m\omega_2$.

(3) If the domain ω is cut by a piecewise smooth surface into two parts ω_1 and ω_2 ($\omega = \omega_1 + \omega_2$), then $m\omega = m\omega_1 + m\omega_2$.

There are sets of three-dimensional measure zero. These are a point, a line segment, a plane rectangle, a smooth or a piecewise smooth surface.

By analogy, we can consider n-dimensional domains ω ($\omega \subset R_n$) with a piecewise smooth boundary in order to define for them an n-dimensional measure $m\omega > 0$ which possesses properties similar to (1), (2), (3).

A rectangle Δ in R_n is defined as the set of points $\mathbf{x} = (x_1, \ldots, x_n)$ whose coordinates satisfy the inequalities

$$a_j \leqslant x_j \leqslant b_j \quad (j = 1, \ldots, n;\ a_j < b_j).$$

The n-dimensional measure of Δ is defined as the product

$$m\Delta = |\Delta| = (b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n).$$

A smooth surface $S \subset R_n$ is defined as the set of points $\mathbf{x} = (x_1, \ldots, x_n)$, satisfying the equation

$$x_j = f(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),$$
$$(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \overline{g},$$

where j may have one of the values $j = 1, 2, \ldots, n$. Here f is a continuously differentiable function in the closure of some $(n-1)$-dimensional bounded domain g of points $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.

By definition, a piecewise smooth surface in R_n consists of a finite number of smooth pieces (surfaces) which either do not intersect or intersect only along their boundaries.

Let us repeat the definition of the multiple integral without resorting to particular problems of geometrical or physical character.

Let in an n-dimensional space R_n there be given a bounded domain in Ω with a piecewise smooth boundary Γ ($\overline{\Omega} = \Omega + \Gamma$) and on Ω (or $\overline{\Omega}$) there be defined a function

$$f(\mathbf{x}) = f(x_1, \ldots, x_n).$$

We cut $\overline{\Omega}$ into Ω_j parts which either do not intersect or intersect only along their boundaries, the latter being regarded as piecewise smooth. For the sake of brevity, we shall say that we have carried out partition ρ of the set Ω.

Now we choose an arbitrary point $\xi^j = (\xi_1^j, \ldots, \xi_n^j)$ ($\xi^j \in \Omega_j$) in each part Ω_j and form the sum

$$S_\rho(f) = \sum_{j=1}^{N} f(\xi^j)\, m\Omega_j,$$

which will be called the *Riemann integral sum* of the function f corresponding to the partition ρ.

The limit of the sum

$$\lim_{\max d(\Omega_j)\to 0} S_\rho(f) = \lim_{\max d(\Omega_j)\to 0} \sum_{j=1}^{N} f(\xi^j)\, m\Omega_j$$

$$= \int_{\Omega} f(\mathbf{x})\, d\mathbf{x} = \int \ldots \int_{\Omega} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n, \quad (5)$$

when the maximum diameter of the subsets Ω_j tends to zero, is called the *multiple integral of f on Ω* (or over Ω).

We should like to stress that limit (5) is termed the multiple integral of f if it is independent both of the choice of points ξ^j in Ω_j and the methods of partition ρ of the domain Ω.

Let us make some remarks.

Remark 1. It makes no difference whether limit (5) is computed for the domain Ω or for its closure $\overline{\Omega}$, since $\overline{\Omega} = \Omega + \Gamma$, where Γ is the boundary of Ω which was assumed to be piecewise smooth. And we know that such a boundary has an n-dimensional measure zero ($m\Gamma = 0$, see the next section).

Remark 2. If limit (5), i.e. the multiple integral $\int_{\Omega} f\, d\Omega$, exists, then the function $f(\mathbf{x})$ is bounded on $\overline{\Omega}$ ($|f(\mathbf{x})| \leqslant$

$\leqslant M$). This is proved just in the same way as in the case of a one-dimensional definite integral.

Remark 3. If $\max d(\Omega_j) \to 0$, then the sum of measures of the parts Ω_j which directly adjoin the piecewise smooth boundary Γ also tends to zero

$$\sum'' m\Omega_j \to 0.$$

Here $\sum''$ symbolyzes that the sum is extended to the parts Ω_j which are adjacent to Γ.

For instance, if the domain Ω is cut into parts with the aid of a rectangular network, as is shown in Fig. 27, then the corresponding partition can be written in the form

$$\Omega = \sum' \Omega_j + \sum'' \Omega_j,$$

where the sum $\sum'$ is extended to the entire squares (contained in Ω), and the sum $\sum''$ to the squares not entirely contained in Ω.

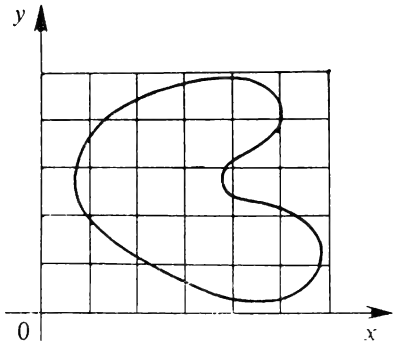

Fig. 27.

It is an important thing that the measure of the second sum tends to zero as the diameter of the diagonal of the network squares tends to zero:

$$\sum'' m\Omega_j \xrightarrow[\max d(\Omega_j)\to 0]{} 0.$$

Remark 4. It follows from the previous remarks that

$$\left|\sum'' f(\xi^j)\, m\Omega_j\right| \leqslant \sum'' M \cdot m\Omega_j = M \sum'' m\Omega_j \xrightarrow[\max d(\Omega_j)\to 0]{} 0.$$

This shows that integral (5) can be defined in the same way as the limit of the sum

$$\lim_{\max d(\Omega_j)\to 0} \sum' f(\xi^j)\, m\Omega_j = \int_{\Omega} f(\mathbf{x})\, d\mathbf{x},$$

extended only to the partition parts Ω_j which do not adjoin Γ.

We have not proved Remarks 1 to 4 formally, since they readily follow from the next section.

Sec. 2.2. Elements of the Theory of the Jordan Measure

We shall confine ourselves to considering two-dimensional sets in the plane with a system of rectangular coordinates (x, y) introduced.

Let us take a natural number N and two systems of straight lines

$$x = kh \quad (k = 0, \pm 1, \pm 2, \ldots),$$
$$y = lh \ (l = 0, \pm 1, \pm 2, \ldots), \quad h = 2^{-N},$$

specifying in the plane a rectangular network consisting of squares with sides of length h. Such a network will be called an h-network (shown in Fig. 28). It is evident that when we

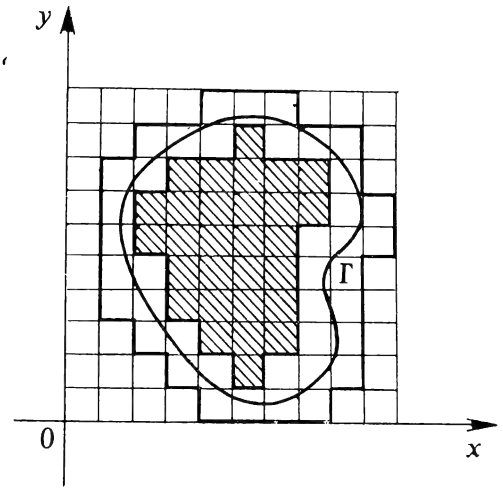

Fig. 28.

pass from N to $N + 1$ each of the squares of the h-network ($h = 2^{-N}$) splits into four congruent squares, the latter forming an $h = 2^{-N-1}$-network.

Let us take in a plane an arbitrary bounded set Ω and introduce two sets $\underline{\Omega}_N$ and $\overline{\Omega}_N$ for the given N. The first of them $\underline{\Omega}_N$ is the set-theoretic sum of the squares of the h-network ($h = 2^{-N}$) each of which is entirely contained in Ω (the hatched portion of the area shown in Fig. 28). $\underline{\Omega}_N$ will

be called the *interior figure of the set* Ω (determined by the given h-network). It may happen that $\underline{\Omega}_N$ is an empty set, i.e. there is not a single square which would entirely belong to Ω. This takes place, for instance, if Ω is a set consisting of a finite number of points or if it is a piece of a smooth curve.

The second set $\overline{\Omega}_N$ is called the *exterior figure of the set* Ω (determined by the given h-network). It is the sum of the squares contained in the h-network each of which contains at least one point of the set Ω.

It is evident that

$$\underline{\Omega}_N \subset \Omega \subset \overline{\Omega}_N,$$

and the areas of the figures $\underline{\Omega}_N$, $\overline{\Omega}_N$, which will be denoted by $|\underline{\Omega}_N|$, $|\overline{\Omega}_N|$, satisfy the inequality

$$|\underline{\Omega}_N| \leqslant |\overline{\Omega}_N| \quad (N = 1, 2, \ldots).$$

If $\underline{\Omega}_N$ is an empty set, then $|\underline{\Omega}_N| = 0$.

It is easily seen that

$$\underline{\Omega}_1 \subset \underline{\Omega}_2 \subset \underline{\Omega}_3 \subset \ldots \subset \Omega \subset \ldots \subset \overline{\Omega}_3 \subset \overline{\Omega}_2 \subset \overline{\Omega}_1,$$

whence

$$|\underline{\Omega}_1| \leqslant |\underline{\Omega}_2| \leqslant |\underline{\Omega}_3| \leqslant \ldots \leqslant |\overline{\Omega}_3| \leqslant |\overline{\Omega}_2| \leqslant |\overline{\Omega}_1|$$

Hence,

$$|\underline{\Omega}_N| \leqslant |\overline{\Omega}_M|,$$

whatever the natural numbers N and M may be.

If M is fixed, then, as N increases without bound, the numbers $|\underline{\Omega}_N|$ do not decrease remaining not greater than the number $|\overline{\Omega}_M|$. This shows that there exists the limit

$$\lim_{N\to\infty} |\underline{\Omega}_N| \leqslant |\overline{\Omega}_M|.$$

It is called the *interior Jordan content* (*measure*) of the set Ω and is denoted:

$$\lim_{N\to\infty} |\underline{\Omega}_N| = m_i\Omega.$$

This is quite a definite number independent of N. Hence, we have obtained the inequality

$$m_i\Omega \leqslant |\overline{\Omega}_M| \quad (M = 1, 2, \ldots),$$

where the numbers $|\overline{\Omega}_M|$ do not increase monotonically as M increases unboundedly. But then there exists the limit

$$\lim_{M\to\infty} |\overline{\Omega}_M| \geqslant m_i\Omega,$$

which is called the *exterior Jordan content (measure)* of the set Ω and is denoted by $m_e\Omega$.

Thus, we have proved that an *arbitrary two-dimensional bounded set* Ω *possesses the interior and exterior Jordan contents (measures)* $m_i\Omega$ *and* $m_e\Omega$. *These are nonnegative numbers satisfying the inequality* $m_i\Omega \leqslant m_e\Omega$.

If a set Ω is such that an equality takes place, then Ω is said to be *Jordan measurable*, and the number

$$m\Omega = m_i\Omega = m_e\Omega$$

is called the *(two-dimensional) Jordan* content (measure) of* Ω.

For the sake of brevity, we shall often omit the adjective "Jordan". For the same reason, in this section where we shall deal only with the Jordan two-dimensional measure we shall sometimes omit the adjective "two-dimensional".

Thus, a set Ω is Jordan measurable if the following holds:

$$\lim_{N\to\infty} |\underline{\Omega}_N| = \lim_{N\to\infty} |\overline{\Omega}_N|. \tag{1}$$

Let us denote by Γ the boundary of the set Ω ($\Gamma = \partial\Omega$). To obtain the number of the network squares covering Γ or, as we shall say, in order to obtain the figure covering Γ (see Fig. 28), we have to subtract (in the set-theoretic sense) the figure $\underline{\Omega}_N$ from the figure $\overline{\Omega}_N$ and to close the set thus obtained

$$\overline{\Gamma}_N = \overline{\overline{\Omega}_N \setminus \underline{\Omega}_N}.$$

* In modern mathematics another measure, viz. the Lebesgue measure, is also of great importance. Lebesgue, Henri Leon (1875-1941). French mathematician.

Obviously, the area (two-dimensional measure) $\overline{\Gamma}_N$ is equal to

$$|\overline{\Gamma}_N| = |\overline{\Omega}_N| - |\underline{\Omega}_N|.$$

It follows from (1) that

$$\lim_{N\to\infty} |\overline{\Gamma}_N| = \lim_{N\to\infty} |\overline{\Omega}_N| - \lim_{N\to\infty} |\underline{\Omega}_N| = 0. \tag{2}$$

Conversely, from the equality

$$\lim_{N\to\infty} |\overline{\Gamma}_N| = 0, \tag{3}$$

taking into account that the limits $\lim\limits_{N\to\infty} |\overline{\Omega}_N|$ and $\lim\limits_{N\to\infty} |\underline{\Omega}_N|$ exist, there follows equality (1), i.e. the measurability of Ω.

Note that limit (3) is the exterior measure of Γ, i.e.

$$m_e\Gamma = 0.$$

But $0 \leqslant m_i\Gamma \leqslant m_e\Gamma$, therefore also

$$m_i\Gamma = m_e\Gamma = 0.$$

We have proved the following important statement: *for a set Ω to be Jordan measurable, it is necessary and sufficient that its boundary Γ be of Jordan measure zero* ($m\Gamma = 0$).

It will be shown below that a *piecewise smooth curve has a two-dimensional measure zero.* But then the *domain Ω, having a piecewise smooth boundary, is Jordan measurable.*

Let us consider some examples.

Example 1. The set Ω consisting of a single point possesses two-dimensional measure zero ($m\Omega = 0$). The point can belong at most to four squares of the h-network; their total area tends to zero as $N \to \infty$ and, consequently, $m_e\Omega = 0$, but $0 \leqslant m_i\Omega \leqslant m_e\Omega$, therefore $m_i\Omega = m_e\Omega = m\Omega = 0$.

Example 2. The continuous curve Γ (Fig. 29) $y = f(x)$ ($a \leqslant x \leqslant b$) has two-dimensional measure zero ($m\Gamma = 0$).

Indeed, since f is uniformly continuous on $[a, b]$, for any $\varepsilon > 0$ there can be found $\delta > 0$ such that $|f(x'') - f(x')| < \varepsilon$ for all $x', x'' \in [a, b]$, satisfying the inequality $|x' - x''| < \delta$. The found $\delta > 0$ can be reduced

at our will. We shall suppose that $\delta < \varepsilon$. Let us take an h-network with

$$h = 2^{-N} < \delta$$

and consider a column made up of the squares containing the points of Γ. Its height does not exceed $\varepsilon + 2h$ (for

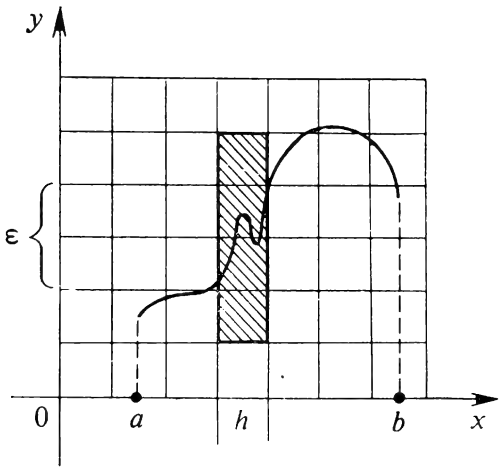

Fig. 29.

$\varepsilon = 2h$, the column hatched in Fig. 29 consists of four squares of the h-network containing the points of Γ and $\varepsilon + 2h = 4h$), and its area does not exceed $(\varepsilon + 2h)\, h$. The total area of the areas covering Γ does not exceed

$$(\varepsilon + 2h)\, h \cdot \frac{K}{h} = (\varepsilon + 2h)\, K \leqslant 3\varepsilon K,$$

where K is the length of the closed interval containing the closed interval $[a, b]$.

This shows that the total area $|\overline{\Gamma}_N|$ of the squares covering the curve Γ, for a sufficiently large N, can be made less than a preassigned arbitrarily small positive number, and, consequently the exterior measure of Γ, and the more so its interior measure, is equal to zero. But then

$$m\Gamma = 0.$$

Since the *sum of a finite number of sets having measure zero* obviously *possesses zero measure*, it follows from Example 1

that the *two-dimensional measure of the set consisting of a finite number of points is equal to zero.*

Further, it follows from Example 2 that the *smooth curve*

$$x = \varphi(t), \quad y = \psi(t) \quad (a \leqslant t \leqslant b) \tag{4}$$

has a zero measure (Fig. 30).

The matter is that if Γ is a smooth curve, then the closed interval $[a, b]$ can be divided into a finite number of subintervals by the points

$$a = t_0 < t_1 < \ldots < t_n = b$$

so that on each subinterval $[t_j, t_{j+1}]$ one of the two equations (4) can be solved with respect to t and substituted into

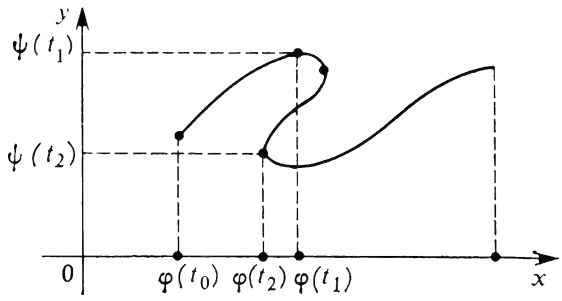

Fig. 30.

the other. As a result, we obtain that the corresponding portion of the curve Γ (Γ_j) is described either by an equation of the form

$$y = f(x), \quad x \in [c, d],$$

or by an equation of the form

$$x = g(y), \quad y \in [p, q],$$

where the functions f and g are continuous on the corresponding subintervals. But then, as we know from Example 2,

$$m\Gamma_j = 0 \quad (j = 1, \ldots, r).$$

Therefore, since Γ is the sum of a finite number of pieces Γ_j

$$\Gamma = \sum_{j=1}^{r} \Gamma_j,$$

each of which has a zero measure, then $m\Gamma = 0$.

Note that if *two sets* Ω_1 *and* Ω_2 *are measurable, then their sum* $\Omega_1 + \Omega_2$, *difference* $\Omega_1 \setminus \Omega_2$, *and intersection* $\Omega_1\Omega_2 = \Omega_1 \cap \Omega_2$ *are also measurable.*

Indeed, let us denote by $\Gamma(E)$ the boundary of the set E. Figure 31 depicts two sets Ω_1 and Ω_2. It is obvious that

$$\left.\begin{aligned} \Gamma(\Omega_1 + \Omega_2) &\subset \Gamma(\Omega_1) + \Gamma(\Omega_2), \\ \Gamma(\Omega_1 \setminus \Omega_2) &\subset \Gamma(\Omega_1) + \Gamma(\Omega_2), \\ \Gamma(\Omega_1\Omega_2) &\subset \Gamma(\Omega_1) + \Gamma(\Omega_2). \end{aligned}\right\} \tag{5}$$

If Ω_1 and Ω_2 are measurable, we have $m\Gamma(\Omega_1) = 0$, $m\Gamma(\Omega_2) = 0$, but then the measures of the left-hand members of (5) are also equal to zero which shows that the sets $\Omega_1 + \Omega_2$, $\Omega_1 \setminus \Omega_2$, $\Omega_1\Omega_2$ are measurable.

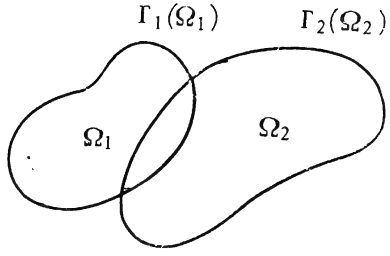

Fig. 31.

Here we have taken advantage of the obvious property of measure: if a set ω has measure zero, then any of its subsets has also measure zero.

Finally, if Ω^1 and Ω^2 are two measurable sets which either do not intersect or intersect only along some parts of their boundaries, then

$$m(\Omega^1 + \Omega^2) = m\Omega^1 + m\Omega^2. \tag{6}$$

Indeed, it is obvious that (see Fig. 32)

$$\underline{\Omega}^1_N + \underline{\Omega}^2_N \subset \underline{(\Omega^1 + \Omega^2)}_N \subset \overline{(\Omega^1 + \Omega^2)}_N \subset \overline{\Omega}^1_N + \overline{\Omega}^2_N,$$

and since

$$\lim_{N\to\infty} |\underline{\Omega}^1_N| = \lim_{N\to\infty} |\overline{\Omega}^1_N| = m\Omega^1,$$

$$\lim_{N\to\infty} |\underline{\Omega}^2_N| = \lim_{N\to\infty} |\overline{\Omega}^2_N| = m\Omega^2,$$

then, obviously,

$$\lim_{N\to\infty} |\underline{(\Omega^1 + \Omega^2)}_N| = \lim_{N\to\infty} |\overline{(\Omega^1 + \Omega^2)}_N| = m\Omega^1 + m\Omega^2,$$

which proves (6).

Notice that if a domain Ω is measurable, then its Jordan measure is equal to the measure of its closure

$$m\Omega = m\overline{\Omega}.$$

Indeed, $\overline{\Omega} = \Omega + \Gamma$, where Γ is the boundary of Ω and $m\overline{\Omega} = m\Omega + m\Gamma$, where $m\Gamma = 0$.

Example 3. The set ω consisting of all rational numbers of the closed interval $[0, 1]$ is Jordan unmeasurable: $m_i\omega = 0$, $m_e\omega = 1$.

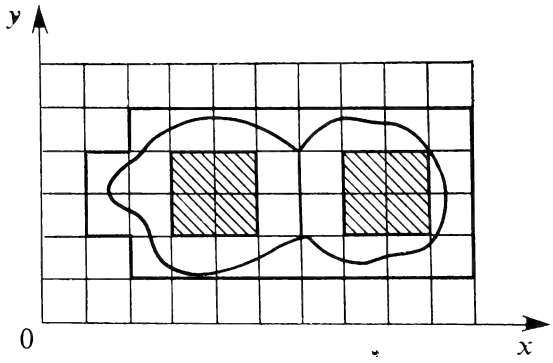

Fig. 32.

In the three-dimensional case the theory of the Jordan measure is analogous. Now we introduce a system of rectangular coordinates (x, y, z) and three families of parallel planes

$$x = kh \quad (k = 0, \pm 1, \pm 2, \ldots),$$
$$y = lh \quad (l = 0, \pm 1, \pm 2, \ldots),$$
$$z = mh \quad (m = 0, \pm 1, \pm 2, \ldots),$$

dividing the space into cubes with edge $h = 2^{-N}$ $(N = 1, 2, \ldots)$. Such partition of space we call, as before, the h-network (three-dimensional).

Let Ω be a bounded set of points belonging to the space under consideration. We denote by $\underline{\Omega}_N$ the interior figure of Ω, i.e. the totality of the cubes entirely belonging to Ω and by $\overline{\Omega}_N$ the exterior figure of the set Ω, i.e. all the cubes each of which contains at least one point of Ω.

We conclude once again that

$$\underline{\Omega}_1 \subset \underline{\Omega}_2 \subset \underline{\Omega}_3 \subset \ldots \subset \Omega,$$
$$\overline{\Omega}_1 \supset \overline{\Omega}_2 \supset \overline{\Omega}_3 \supset \ldots \supset \Omega,$$

whence it follows

$$|\underline{\Omega}_1| \leqslant |\underline{\Omega}_2| \leqslant |\underline{\Omega}_3| \leqslant \ldots,$$
$$|\overline{\Omega}_1| \geqslant |\overline{\Omega}_2| \geqslant |\overline{\Omega}_3| \geqslant \ldots$$

and

$$|\underline{\Omega}_N| \leqslant |\overline{\Omega}_M|,$$

for any natural N and M. Hence there follows the existence of the limits

$$\lim_{N\to\infty} |\underline{\Omega}_N| \leqslant \lim_{N\to\infty} |\overline{\Omega}_N|.$$

The first limit in this inequality is called the interior (three-dimensional) measure of Ω:

$$m_i\Omega = \lim_{N\to\infty} |\underline{\Omega}_N|,$$

and the second limit its exterior measure:

$$m_e\Omega = \lim_{N\to\infty} |\overline{\Omega}_N|.$$

Hence,

$$m_i\Omega \leqslant m_e\Omega.$$

If

$$m_i\Omega = m_e\Omega = m\Omega,$$

then the set Ω is said to be *Jordan measurable in the three-dimensional sense*, and the number $m\Omega$ is called its *three-dimensional measure.*

Reasoning much in the same manner as it was in connection with equalities (1), (2), (3), we prove that the set is measurable in the three-dimensional sense if and only if its boundary possesses three-dimensional measure zero.

We are not going to formulate other properties of sets measurable in the three-dimensional sense. They are similar to the aforegoing properties of sets measurable in the two-dimensional sense. Let us explain only that a piecewise smooth surface has three-dimensional measure zero. Such

a surface consists of a finite number of pieces S which do not intersect or intersect only along their boundaries each of which is determined by the equation (in the newly denoted coordinates)

$$z = f(x, y) \quad ((x, y) \in \bar{g}),$$

where $\bar{g}$ is a closure of some domain bounded in the xy-plane.

Let us take $\varepsilon > 0$ and choose $\delta > 0$ so that

$$| f(x', y') - f(x'', y'') | < \varepsilon$$

for all points (x', y'), $(x'', y'') \in \bar{g}$, found at a distance

$$| (x', y') - (x'', y'') | < \delta$$

from one another.

Assuming $\delta < \varepsilon$, we take an h-network with $h = 2^{-N} < \delta$ and consider a column constructed from cubes containing some points of S. Its altitude does not exceed $\varepsilon + 2h$, and the volume does not exceed $(\varepsilon + 2h)\, h^2$. The total volume of all such columns covering S does not exceed

$$(\varepsilon + 2h)\, h^2 \cdot \frac{K}{h^2} = (\varepsilon + 2h)\, K < 3\varepsilon K. \tag{7}$$

Here K is the area of the square Δ with integral coordinates of its vertices covering the set $\bar{g}$. Since the vertices of Δ have integral coordinates, they are found on the planes of the network under consideration. The right-hand member of (7) may be taken arbitrarily small which proves that the three-dimensional measure $mS = 0$.

By analogy, we can introduce the notion of n-dimensional measure for the sets of the space R_N and show that a smooth surface in R_N has n-dimensional measure zero.

Sec. 2.3. Properties of Multiple Integrals. Existence Theorems

In our further considerations, Ω, Ω_1, Ω_2 will denote domains with a piecewise smooth boundary (although they may be regarded as arbitrarily Jordan measurable).

The following equality holds

$$\int_{\Omega} d\mathbf{x} = \int_{\Omega} 1 \cdot d\mathbf{x} = m\Omega. \tag{1}$$

This equality is obvious. To compute integral (1), it is necessary to cut the domain Ω by piecewise smooth surfaces into parts

$$\Omega = \sum_{k=1}^{N} \Omega_k,$$

which either do not intersect or intersect only along their boundaries (as in Fig. 33) and take into account that

$$\sum_{k=1}^{N} 1 \cdot m\Omega_k = \sum_{k=1}^{N} m\Omega_k = m\Omega.$$

But then

$$\lim_{\max d(\Omega_k)\to 0} \sum_{k=1}^{N} m\Omega_k = m\Omega.$$

Using formula (1) in the two-dimensional case we compute the area Ω, in the three-dimensional case the volume Ω. And in the n-dimensional case formula (1) yields an n-dimensional measure Ω.

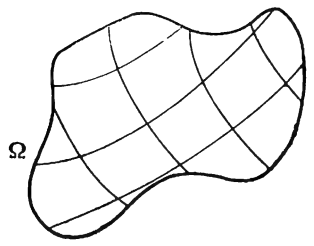

Fig. 33.

We shall see below that the computation of integral (1) and of the more general integral $\int_{\Omega} f\,d\mathbf{x}$ can be reduced to successive computation of some one-dimensional integrals (see Sec. 2.4).

We shall assume below that for the functions $f(\mathbf{x})$, $\varphi(\mathbf{x})$, $|f(\mathbf{x})|$, to be dealt with, the integrals under consideration do exist.

There takes place the following equality

$$\int_{\Omega} [Af(\mathbf{x}) + B\varphi(\mathbf{x})]\,d\mathbf{x} = A\int_{\Omega} f(\mathbf{x})\,d\mathbf{x} + B\int_{\Omega} \varphi(\mathbf{x})\,d\mathbf{x}, \quad (2)$$

where A and B are constants.

If a domain Ω with a piecewise smooth boundary is cut into measurable parts Ω_1, Ω_2 ($\overline{\Omega} = \overline{\Omega_1} + \overline{\Omega_2}$), then

$$\int_{\Omega} f\,d\mathbf{x} = \int_{\Omega_1} f\,d\mathbf{x} + \int_{\Omega_2} f\,d\mathbf{x}. \quad (3)$$

If

$$f(\mathbf{x}) \leqslant \varphi(\mathbf{x}) \quad (\mathbf{x} \in \Omega), \tag{4}$$

then

$$\int_{\Omega} f(\mathbf{x})\, d\mathbf{x} \leqslant \int_{\Omega} \varphi(\mathbf{x})\, d\mathbf{x}. \tag{5}$$

But then, taking into account that

$$-|f(\mathbf{x})| \leqslant f(\mathbf{x}) \leqslant |f(\mathbf{x})|,$$

we obtain, by virtue of (2) (for $A = -1$, $B = 0$), that

$$-\int_{\Omega} |f(\mathbf{x})|\, d\mathbf{x} \leqslant \int_{\Omega} f(\mathbf{x})\, d\mathbf{x} \leqslant \int_{\Omega} |f(\mathbf{x})|\, d\mathbf{x},$$

i.e.

$$\left| \int_{\Omega} f(\mathbf{x})\, d\mathbf{x} \right| \leqslant \int_{\Omega} |f(\mathbf{x})|\, d\mathbf{x}. \tag{6}$$

In particular, it follows from (6) that if

$$|f(\mathbf{x})| \leqslant M \quad (\mathbf{x} \in \Omega), \tag{7}$$

where M is a constant, then

$$\left| \int_{\Omega} f(\mathbf{x})\, d\mathbf{x} \right| \leqslant \int_{\Omega} M\, d\mathbf{x} = M \int_{\Omega} d\mathbf{x} = M \cdot m\Omega. \tag{8}$$

The following existence theorem holds true (for its proof see Sec. 2.5).

Theorem 1. *If a function $f(\mathbf{x})$ is continuous on the closure $\overline{\Omega}$ of a bounded domain Ω with a piecewise smooth boundary, then it is integrable on Ω, as well as on $\overline{\Omega}$, and*

$$\int_{\Omega} f(\mathbf{x})\, d\mathbf{x} = \int_{\overline{\Omega}} f(\mathbf{x})\, d\mathbf{x}. \tag{9}$$

Note that

$$\int_{\overline{\Omega}} f\, d\mathbf{x} = \int_{\Omega} f\, d\mathbf{x} + \int_{\Gamma} f\, d\mathbf{x} = \int_{\Omega} f\, d\mathbf{x} + 0 = \int_{\Omega} f\, d\mathbf{x},$$

where Γ is the boundary of Ω. By the hypothesis, Γ is a piecewise smooth boundary; its n-dimensional measure is

equal to zero ($m\Gamma = 0$), consequently,

$$\int_{\Gamma} f\, d\mathbf{x} = 0.$$

Let us state a more general existence theorem.

Theorem 2. *If a function $f(\mathbf{x})$ is bounded and continuous everywhere on the closure $\overline{\Omega}$ of a bounded domain Ω with a piecewise smooth boundary except for a finite number of individual points and smooth curves, where it may have discontinuities, then f is integrable on Ω, as well as on $\overline{\Omega}$ and equality* (9) *is fulfilled.*

Note one more theorem.

Theorem 3 (mean-value theorem). *Let a function $f(\mathbf{x})$ be continuous in the closure $\overline{\Omega}$ of a domain Ω which is supposed to be connected*.* Then there is a point $\mathbf{x}^0$ ($\mathbf{x}^0 \in \overline{\Omega}$) *such that the equality*

$$\int_{\Omega} f(\mathbf{x})\, d\mathbf{x} = f(\mathbf{x}^0)\, m\Omega \tag{10}$$

is fulfilled.

Proof. By the hypothesis, the function f is continuous on a closed bounded $\overline{\Omega}$, therefore on $\overline{\Omega}$ there exist points $\mathbf{x}^1$ and $\mathbf{x}^2$ ($\mathbf{x}^1, \mathbf{x}^2 \in \overline{\Omega}$) at which f attains, respectively, its minimum and maximum on $\overline{\Omega}$:

$$f(\mathbf{x}^1) \leqslant f(\mathbf{x}) \leqslant f(\mathbf{x}^2), \quad \forall \mathbf{x} \in \overline{\Omega}.$$

Integrating these inequalities over Ω, we obtain

$$\int_{\Omega} f(\mathbf{x}^1)\, d\mathbf{x} \leqslant \int_{\Omega} f(\mathbf{x})\, d\mathbf{x} \leqslant \int_{\Omega} f(\mathbf{x}^2)\, d\mathbf{x}$$

or

$$f(\mathbf{x}^1)\, m\Omega \leqslant \int_{\Omega} f(\mathbf{x})\, d\mathbf{x} \leqslant f(\mathbf{x}^2)\, m\Omega. \tag{11}$$

Hence it follows that the number

$$C = \frac{1}{m\Omega} \int_{\Omega} f\, d\mathbf{x} \quad (m\Omega \neq 0)$$

* A set E is called *connected* if any pair of its points can be joined by a continuous curve belonging to E.

is found between the least $(f(\mathbf{x}^1))$ and the greatest value $(f(\mathbf{x}^2))$ of the function. Since $\overline{\Omega}$ is a connected set, there exists a continuous curve belonging to $\overline{\Omega}$:

$$\mathbf{r}(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)), \quad t_1 \leqslant t \leqslant t_2,$$

which connects the points $\mathbf{x}^1$ and $\mathbf{x}^2$, i.e. such that

$$\mathbf{r}(t_1) = \mathbf{x}^1, \quad \mathbf{r}(t_2) = \mathbf{x}^2.$$

The function

$$F(t) \equiv f[\mathbf{r}(t)] = f(\varphi_1(t), \ldots, \varphi_n(t))$$

is continuous on the closed interval $[t_1, t_2]$ (as a superposition of continuous functions) and takes on the following respective values at its end points

$$F(t_1) = f(\mathbf{x}^1), \quad F(t_2) = f(\mathbf{x}^2).$$

But then, by the theorem on an intermediate value for the function $F(t)$ of one variable, there exists $t_0 \in [t_1, t_2]$ such that at the point $\mathbf{x}^0 = \mathbf{r}(t_0)$ there takes place the equality

$$F(t_0) = f[\mathbf{r}(t_0)] = f(x^0) = C.$$

We have thus proved (10).

Remark 1. The number $f(\mathbf{x}^0)$ in (10) is called the *mean value* of the continuous function f on the domain Ω.

Sec. 2.4. Integral as a Function of Parameter. Reduction of Multiple Integral to Iterated Integrals

Consider the integral

$$F(x) = \int_c^d f(x, y)\, dy, \tag{1}$$

where the function $f(x, y)$ is defined on the rectangle

$$\Delta = [a, b] \times [c, d], \tag{2}$$

i.e. on the set of points (x, y), where

$$a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d \quad (a < b, \ c < d).$$

Here integration is carried out with respect to the variable y.

But the integrand depends not only on y, but also on x, therefore integral (1) is a function of x ($F(x)$).

Integral (1) is said to be a function of the parameter x.

Theorem 1. *If a function $f(x, y)$ is continuous in a rectangle Δ, then the function $F(x)$ is continuous on the closed interval $[a, b]$.*

Proof. We have

$$F(x+h) - F(x) = \int_c^d [f(x+h, y) - f(x, y)]\, dy$$

$$(x,\ x+h \in [a,\ b]).$$

Since the function f is continuous on a closed bounded set Δ, it is uniformly continuous on Δ. Consequently, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(x+h, y) - f(x, y)| < \frac{\varepsilon}{d-c}$$

for all (x, y), $(x+h, y) \in \Delta$, if only $|h| < \delta$. But then

$$|F(x+h) - F(x)| \leqslant \int_c^d \frac{\varepsilon}{d-c}\, dy = \frac{\varepsilon}{d-c} \cdot (d-c) = \varepsilon,$$

and the theorem has been proved.

Theorem 2. *Under the conditions of Theorem 1 there exists an iterated integral*

$$\int_a^b F(x)\, dx = \int_a^b dx \int_c^d f(x, y)\, dy. \tag{3}$$

Indeed, the function $F(x)$ continuous on the closed interval $[a, b]$ is integrable on this interval.

Theorem 3. *Under the conditions of Theorem 1 the following equalities hold:*

$$\iint_\Delta f(x, y)\, dx\, dy = \int_a^b dx \int_c^d f(x, y)\, dy = \int_c^d dy \int_a^b f(x, y)\, dx. \tag{4}$$

The first member of chain (4) is a multiple integral of a continuous function on a closed set Δ with a piecewise smooth boundary. It is existent (see below Sec. 2.5). The iterated integrals represented by the second and third

members of the chain of equalities (4) also exist (by Theorem 2).

This theorem states the equality of these three integrals. Thus, the computation of a multiple integral has been reduced to computation of one-dimensional integrals separately with respect to x and y.

Proof. Let us divide the sides of Δ into N equal parts:

$$a = x_0 < x_1 < \ldots < x_N = b,$$
$$c = y_0 < y_1 < \ldots < y_N = d,$$

and, through the division points, draw straight lines parallel to the y- and x-axes. As a result of this procedure, Δ has been divided into congruent rectangles Δ_{kl}:

$$\Delta = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \Delta_{kl},$$

$$\Delta_{kl} = \{x_k \leqslant x \leqslant x_{k+1},\ y_l \leqslant y \leqslant y_{l+1}\}$$

and

$$\int_a^b dx \int_c^d f(x, y)\, dy = \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} dx \int_c^d f(x, y)\, dy$$

$$= \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} dx \sum_{l=0}^{N-1} \int_{y_l}^{y_{l+1}} f(x, y)\, dy$$

$$= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \int_{x_k}^{x_{k+1}} dx \int_{y_l}^{y_{l+1}} f(x, y)\, dy$$

$$= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \int_{y_l}^{y_{l+1}} f(\xi_k, y)\, dy\, \Delta x_k = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(\xi_k, \eta_l)\, \Delta x_k\, \Delta y_l \quad (5)$$

$$(x_k \leqslant \xi_k \leqslant x_{k+1},\ y_l \leqslant \eta_l \leqslant y_{l+1}).$$

We have applied the mean-value theorem first to the integral $\int_{x_k}^{x_{k+1}}$ over x of the function $\int_{y_l}^{y_{l+1}} f(x, y)\, dy$ and then to the integral $\int_{y_l}^{y_{l+1}} f(\xi_k, y)\, dy$.

We have proved that the iterated integral in the left-hand side of (5) can be regarded as the integral sum of the multiple integral $\iint\limits_{\Delta} f\,dx\,dy$, corresponding to the partition of Δ into parts Δ_{kl} with the aid of some points $(\xi_k, \eta_l) \in \Delta_{kl}$.

Let us now pass to the limit in equality (5) as $N \to \infty$. Then the left-hand side of (5) is a certain number (independent of N), while the right-hand side tends, as $N \to \infty$, to the multiple integral of f over Δ (if the multiple integral is existent, then any integral sum tends to this integral). Therefore

$$\int\limits_a^b dx \int\limits_c^d f(x, y)\,dy = \iint\limits_{\Delta} f\,dx\,dy,$$

and we have proved the first equality of (4). The equality of the second iterated integral to the multiple integral is proved in a similar way.

Let us consider in the xy-plane the domain Ω bounded by smooth curves (shown in Fig. 34)

$$y = \varphi(x), \quad y = \psi(x) \quad (a \leqslant x \leqslant b),$$

where

$$\varphi(x) \leqslant \psi(x) \quad (a \leqslant x \leqslant b).$$

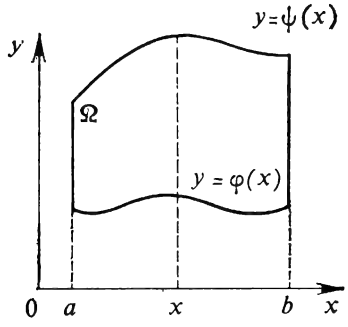

Fig. 34.

Let there be given an arbitrary continuous function $f(x, y)$ in the closure $\overline{\Omega}$ of domain Ω. To compute the multiple integral of $f(x, y)$ over the domain Ω (or, which is the same, over $\overline{\Omega}$), the following method is applied: first the function $f(x, y)$ is integrated with respect to the variable y from $y = \varphi(x)$ to $y = \psi(x)$, regarding x as constant, and then the result thus obtained is integrated with respect to x over the closed interval $[a, b]$:

$$\iint\limits_{\Omega} f(x, y)\,dx\,dy = \int\limits_a^b dx \int\limits_{\varphi(x)}^{\psi(x)} f(x, y)\,dy. \tag{6}$$

For the case when Ω is a rectangle whose sides are parallel to the coordinate axes equality (6) was proved above (see Theorem 3). Using the same arguments, we could prove (6) in the general case.

Example 1. Compute the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1 \quad (a,\ b>0).$$

We define the ellipse both as a set of points (x, y) for which $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1$, and also as a set of points (x, y) for which $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. We shall take a similar liberty by defining the ellipsoid both as a set of points (x, y, z) for which $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1$ and also as a set of points (x, y, z) satisfying the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution. The domain Ω in question is bounded below by the curve

$$y=-\frac{b}{a}\sqrt{a^2-x^2} \quad (-a\leqslant x\leqslant a)$$

and above by the curve

$$y=\frac{b}{a}\sqrt{a^2-x^2} \quad (-a\leqslant x\leqslant a).$$

The required area is equal to

$$m\Omega=\iint\limits_{\Omega} dx\,dy=\int\limits_{-a}^{a} dx \int\limits_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dy=\frac{2b}{a}\int\limits_{-a}^{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{4b}{a}\int\limits_{0}^{a}\sqrt{a^2-x^2}\,dx=(x=a\sin\theta,\ dx=a\cos\theta\,d\theta)$$

$$=4b\int\limits_{0}^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\cos\theta\,d\theta=4ab\int\limits_{0}^{\pi/2}\cos^2\theta\,d\theta$$

$$=4ab\int\limits_{0}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta=2ab\left[\theta+\frac{\sin 2\theta}{\theta}\right]\Bigg|_0^{\pi/2}$$

$$=2ab\cdot\frac{\pi}{2}=\pi ab.$$

Example 2. Compute the area of the figure Ω bounded by a parabola $y = x^2$ and a straight line $y = a^2$ (Fig. 35). We have

$$m\Omega = \iint_{\Omega} dx\,dy = 2\int_0^{a^2} dy \int_a^{\sqrt{y}} dx = 2\int_0^{a^2} \sqrt{y}\,dy = \frac{4}{3}a^3.$$

The method of computing multiple integrals by reducing them to one-dimensional integrals is applicable to spaces of any measures. Let us illustrate this by considering a number of particular examples.

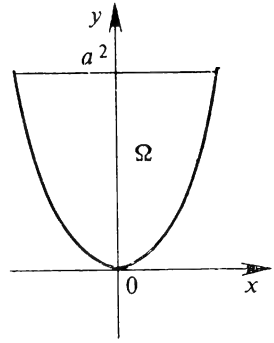

Fig. 35.

Example 3. Compute the volume Ω of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1 \quad (a,\ b,\ c > 0),$$

To this end, let us solve the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

which yields two continuous functions

$$z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

These functions are defined on the set ω of points (x, y) satisfying the inequality

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1.$$

The volume or three-dimensional measure Ω is equal to the integral

$$V = m\Omega = \iiint_{\Omega} 1 \cdot dx\,dy\,dz.$$

To compute this integral, we have to integrate the unit function (equal to 1) with respect to z for any point $(x, y) \in \omega$ from $z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ to $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$. The result, dependent on (x, y) should be then integrated

over all the points $(x, y) \in \omega$:

$$V = \iiint\limits_{\Omega} dx\,dy\,dz = \iint\limits_{\omega} dx\,dy \int\limits_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz$$

$$= 2c \iint\limits_{\omega} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy$$

$$= 2c \int\limits_{-a}^{a} dx \int\limits_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy.$$

The last equality in this chain follows from formula (6). The further computations require only one thing—high integration techniques:

$$V = \frac{8c}{b} \int\limits_{0}^{a} dx \int\limits_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{\frac{b^2}{a^2}(a-x^2)-y^2}\,dy$$

$$= \left(y = \frac{b}{a}\sqrt{a^2-x^2}\sin t\right) = \frac{8c}{b} \int\limits_{0}^{a} dx \int\limits_{0}^{\pi/2} \frac{b^2}{a^2}(a^2-x^2)\cos^2 t\,dt$$

$$= \frac{8bc}{a^2}\,\frac{\pi}{4} \int\limits_{0}^{a} (a^2-x^2)\,dx; \qquad \int\limits_{0}^{\pi/2} \cos^2 t\,dt = \frac{\pi}{4}.$$

Further

$$V = \frac{2\pi bc}{a^2}\left[a^2x - \frac{x^2}{3}\right]\Big|_0^a = \frac{4}{3}\pi abc.$$

Example 4. Find the volume of the portion of the paraboloid of revolution Ω: $x^2 + y^2 \leqslant z \leqslant a^2$ (Fig. 36).

The same as in Example 3, we have

$$m\Omega = \iiint\limits_{\Omega} dx\,dy\,dz = \iint\limits_{\omega} dx\,dy \int\limits_{x^2+y^2}^{a^2} dz,$$

where ω is the set of points (x, y) satisfying the inequality $x^2 + y^2 \leqslant a^2$, which describes a circle of radius a. Further

we have

$$m\Omega = \iint\limits_{\omega} (a^2 - x^2 - y^2)\, dx\, dy = \int\limits_{-a}^{a} dx \int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)\, dy$$

$$= 4 \int\limits_0^a dx \int\limits_0^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)\, dy = \frac{8}{3} \int\limits_0^a (a^2 - x^2)^{3/2}\, dx$$

$$= (x = a \sin t) = \frac{8a^4}{3} \int\limits_0^{\pi/2} \cos^4 t\, dt = \frac{8a^4}{3} \int\limits_0^{\pi/2} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \frac{\pi a^4}{3} + \frac{2a^4}{3} \int\limits_0^{\pi/2} \cos^2 2t\, dt = \frac{\pi a^4}{3} + \frac{a^4}{3} \int\limits_0^{\pi/2} (1 + \cos 4t)\, dt$$

$$= \frac{\pi a^4}{3} + \frac{\pi a^4}{6} = \frac{\pi a^4}{2}.$$

Example 5. Find the volume of the solid Ω bounded by the surfaces of paraboloids of revolution $x^2 + y^2 = z$, $z = 2(x^2 + y^2)$ and cylindrical surfaces $y = \sqrt{x}$, $y = x^2$ (see Fig. 37).

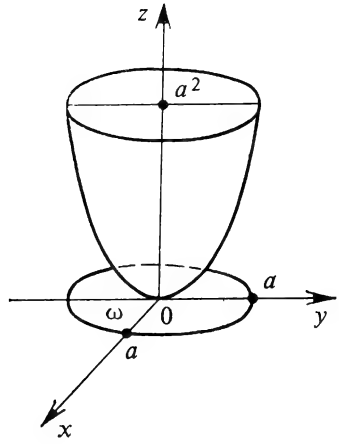

Fig. 36.

Our solid represents a "parabolic shoe" cut by the cylindrical surfaces $y = \sqrt{x}$ and $y = x^2$ between the paraboloids

of revolution. The "shoe" is bounded by a piece of the surface $z = x^2 + y^2$ from below and by a piece of the surface of the paraboloid of revolution $z = 2(x^2 + y^2)$ from above.

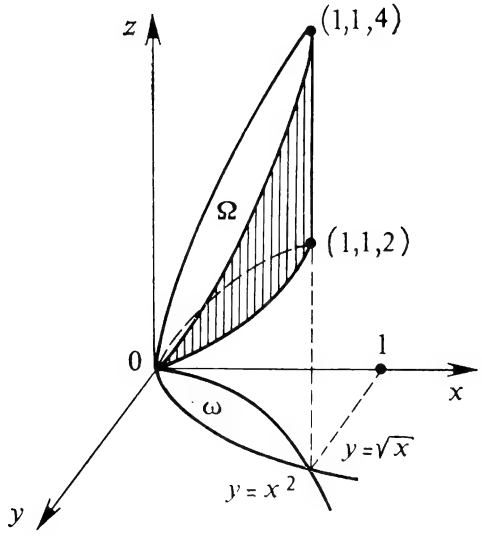

Fig. 37.

The projection of this solid Ω on the xy-plane yields the set ω consisting of points (x, y) whose coordinates satisfy the inequalities: $0 \leqslant x \leqslant 1$, $x^2 \leqslant y \leqslant \sqrt{x}$.

Therefore

$$m\Omega = \iiint_{\Omega} dx\,dy\,dz = \iint_{\omega} dx\,dy \int_{x^2+y^2}^{2(x^2+y^2)} dz$$

$$= \iint_{\omega} (x^2+y^2)\,dx\,dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x^2+y^2)\,dy$$

$$= \int_0^1 \left[x^2(\sqrt{x} - x^2) + \frac{1}{3}(x^{3/2} - x^6) \right] dx$$

$$= \int_0^1 \left(x^{5/2} - x^4 + \frac{x^{3/2}}{3} - \frac{x^6}{3} \right) dx = \frac{2}{7} - \frac{1}{5} + \frac{2}{15} - \frac{1}{21} = \frac{6}{35}.$$

Let us return to integral (1) dependent on a parameter.

Theorem 4. *If a function $f(x, y)$ is continuous and has a continuous derivative f'_x in the rectangle*

$$\Delta = [a, b] \times [c, d],$$

then the function

$$F(x) = \int_c^d f(x, y)\, dy$$

has a derivative on the interval $[a, b]$, and

$$F'(x) = \int_c^d f'_x(x, y)\, dy. \tag{7}$$

Proof. We must prove that

$$\lim_{\Delta x \to 0} \frac{\Delta F(x)}{\Delta x} - \int_c^d f'_x(x, y)\, dy. \tag{8}$$

Applying the equality (Newton-Leibniz formula)

$$\varphi(x+h) - \varphi(x) = \int_0^h \varphi'(x+u)\, du,$$

we have

$$\Lambda = \left| \frac{\Delta F(x)}{\Delta x} - \int_c^d f'_x(x, y)\, dy \right|$$

$$= \left| \int_c^d \frac{\Delta_x f(x, y)}{\Delta x}\, dy - \int_c^d f'_x(x, y)\, dy \right|$$

$$= \left| \int_c^d \left(\frac{1}{\Delta x} \int_0^{\Delta x} f'_x(x+u, y)\, du \right) dy - \int_c^d f'_x(x, y)\, dy \right|$$

$$= \left| \int_c^d \frac{1}{\Delta x} \int_0^{\Delta x} [f'_x(x+u, y) - f'_x(x, y)]\, du\, dy \right|.$$

Since $f'_x(x, y)$ is continuous on Δ, it will also be uniformly continuous on Δ, therefore

$$| f'_x(x, y) - f'_x(x+u, y) | < \varepsilon$$

as soon as $|u| \leqslant |\Delta x| < \delta$ for any points $(x, y) \in \Delta$. On this basis

$$\Lambda \leqslant \int_c^d \frac{1}{|\Delta x|} \left| \int_0^{\Delta x} \varepsilon \, du \right| dy = \varepsilon (d - c)$$

for $|\Delta x| < \delta$. Thus, equality (8), and together with it (7), have been proved.

Let us now consider a more general (as compared with (1)) integral

$$F(x) = \int_{\varphi(x)}^{\psi(x)} f(x, y) \, dy, \tag{9}$$

where $\varphi(x)$ and $\psi(x)$ are functions continuous on $[a, b]$, satisfying the inequalities

$$\varphi(x) \leqslant \psi(x) \quad (a \leqslant x \leqslant b),$$

and $f(x, y)$ is continuous for all the points (x, y) satisfying the inequalities

$$a \leqslant x \leqslant b, \quad \varphi(x) \leqslant y \leqslant \psi(x).$$

Let us show that under the mentioned conditions the function $F(x)$ is continuous on $[a, b]$.

Indeed, let us replace in integral (9) the variable of integration y by another variable u with the aid of the equality

$$y = \varphi(x) + u[\psi(x) - \varphi(x)] \quad (a \leqslant x \leqslant b). \tag{10}$$

For $u = 0$ $y = \varphi(x)$, for $u = 1$ $y = \psi(x)$, $dy = [\psi(x) - \varphi(x)] \, du$ (x is fixed, and we regard it as constant when substituting the variable). Hence,

$$F(x) = [\psi(x) - \varphi(x)] \int_0^1 f(x, \varphi(x) + u[\psi(x) - \varphi(x)]) \, du. \tag{11}$$

Here the factor $\psi(x) - \varphi(x)$ is a continuous function of $x \in [a, b]$ and under the sign of integral stands a continuous function of the point (x, u) belonging to the rectangle $[a, b] \times [0, 1]$. Therefore, according to Theorem 1, the integral itself is a continuous function on $[a, b]$. The

function $F(x)$ will be continuous on $[a, b]$ as a product of two continuous functions.

Theorem 5. *If a function $f(x, y)$ is continuous together with its partial derivative f'_x on*

$$\Omega = \{a \leqslant x \leqslant b, \quad \varphi(x) \leqslant y \leqslant \psi(x)\},$$

and the functions φ, ψ, φ', ψ' are continuous on $[a, b]$, then the function (9) has a derivative computed by the formula

$$F'(x) = \int_{\varphi(x)}^{\psi(x)} f'_x(x, y)\, dy + \psi'(x) f(x, \psi(x)) - \varphi'(x) f(x, \varphi(x)). \quad (12)$$

Proof. We introduce an auxiliary function

$$I(x, u, v) = \int_u^v f(x, y)\, dy,$$

defined on the set $\{a \leqslant x \leqslant b, \varphi(x) \leqslant u \leqslant v \leqslant \psi(x)\}$. According to Theorem 4,

$$\frac{\partial I}{\partial x} = \int_u^v f'_x(x, y)\, dy.$$

Applying the rule of differentiation of an integral with a variable limit of integration, we have

$$\frac{\partial I}{\partial v} = f(x, v), \quad \frac{\partial I}{\partial u} = -f(x, u).$$

Further, $F(x) = I[x, \varphi(x), \psi(x)]$, therefore, according to the rule for differentiation of a composite function, we obtain

$$\frac{dF}{dx} = \frac{\partial I[x, \varphi(x), \psi(x)]}{\partial x} + \frac{\partial I[\ldots]}{\partial u}\frac{d\varphi}{dx} + \frac{\partial I[\ldots]}{\partial v}\frac{d\psi}{dx}$$

$$= \int_{\varphi(x)}^{\psi(x)} f'_x(x, y)\, dy - f(x, \varphi(x))\varphi'(x) + f(x, \psi(x))\psi'(x),$$

which was required to be proved.

Example 6. Find the derivative of the function

$$F(x) = \int_{x^2}^{x} \sin(x^2 + y^2)\, dy.$$

According to (12), we obtain

$$F'(x) = \int_{x^2}^{x} \cos(x^2 + y^2) \cdot 2x\, dy + 1 \cdot \sin(x^2 + x^2) - 2x \sin(x^2 + x^4).$$

Sec. 2.5. Proof of Existence of Integral of a Continuous Function

Let there be given a continuous function $f(\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_n)$ on a closed bounded set $\overline{\Omega}$ in the space R_n. We hold that $m\Omega > 0$. Let us carry out two partitions ρ and ρ' of the domain Ω into measurable parts which either do not intersect or intersect only along their boundaries:

$$\Omega = \bigcup_{k=1}^{N} \Omega_k, \quad \Omega = \bigcup_{l=1}^{M} \Omega'_l$$

and determine the integral sums corresponding to them:

$$S_\rho = \sum_{k=1}^{N} f(\xi^k)\, m\Omega_k \quad (\xi^k = (\xi_1^k, \ldots, \xi_n^k) \in \Omega_k),$$

$$S_{\rho'} = \sum_{l=1}^{M} f(\eta^l)\, m\Omega'_l \quad (\eta^l = (\eta_1^l, \ldots, \eta_n^l) \in \Omega'_l).$$

The intersection of Ω_k and Ω'_l will be denoted by

$$\Omega_{kl} = \Omega_k \Omega'_l.$$

Obviously,

$$S_\rho = \sum_{k=1}^{N} f(\xi^k) \sum_{l=1}^{M} m\Omega_{kl} = \sum_{k=1}^{N} \sum_{l=1}^{M} f(\xi^k)\, m\Omega_{kl},$$

$$S_{\rho'} = \sum_{l=1}^{M} f(\eta^l) \sum_{k=1}^{N} m\Omega_{kl} = \sum_{k=1}^{N} \sum_{l=1}^{M} f(\eta^l)\, m\Omega_{kl}.$$

Therefore

$$| S_\rho - S_{\rho'} | \leqslant \sum_{k=1}^{N} \sum_{l=1}^{M} | f(\xi^k) - f(\eta^l) | \, m\Omega_{kl}$$

$$\leqslant \sum\nolimits' | f(\xi^k) - f(\eta^l) | \, m\Omega_{kl},$$

where the sum $\sum'$ is extended to non-empty sets Ω_{kl}, since the measure of an empty set is all the same equal to zero.

Since the function $f(\mathbf{x})$ is continuous on the closed bounded set $\bar{\Omega}$, it is uniformly continuous on it; therefore for any $\varepsilon > 0$ we can indicate $\delta > 0$ such that

$$| f(\mathbf{x}') - f(\mathbf{x}'') | < \frac{\varepsilon}{2m\Omega}$$

for any $\mathbf{x}'$, $\mathbf{x}'' \in \bar{\Omega}$, satisfying the inequalities

$$| \mathbf{x}' - \mathbf{x}'' | < \delta.$$

We shall suppose that the partitions ρ and ρ' are such that the diameters of their partial sets are less than δ:

$$d(\Omega_k) < \delta, \quad d(\Omega'_l) < \delta. \tag{1}$$

But then on the non-empty set Ω_{kl}

$$| f(\xi^k) - f(\eta^l) | = | f(\xi^k) - f(\lambda^{kl}) + f(\lambda^{kl}) - f(\eta^l) |$$

$$\leqslant | f(\xi^k) - f(\lambda^{kl}) | + | f(\lambda^{kl}) - f(\eta^l) | < \frac{\varepsilon}{2m\Omega} + \frac{\varepsilon}{2m\Omega} = \frac{\varepsilon}{m\Omega}$$

where

$$\lambda^{kl} = (\lambda_1^{kl}, \ldots, \lambda_n^{kl})$$

there is some point belonging to Ω_{kl}. But then, with condition (1) fulfilled,

$$| S_\rho - S_{\rho'} | < \sum\nolimits' \frac{\varepsilon}{m\Omega} m\Omega_{kl} = \frac{\varepsilon}{m\Omega} \sum\nolimits' m\Omega_{kl} = \frac{\varepsilon}{m\Omega} \cdot m\Omega = \varepsilon, \tag{2}$$

which, as it can be proved, involves existence of an integral of f over $\bar{\Omega}$.

Indeed, let us take a sequence of partitions

$$\rho^l: \sum_{j=1}^{Ne} \Omega_j^l = \Omega \quad (l = 1, 2, \ldots)$$

such that

$$\max_j d(\Omega_j^l) \xrightarrow[l\to\infty]{} 0. \tag{3}$$

On the basis of properties (1) and (2), for any $\varepsilon > 0$ there can be found N such that

$$|S_{\rho l} - S_{\rho m}| < \varepsilon \quad (l, m > N).$$

But then, according to Cauchy's criterion, there exists the limit

$$\lim_{l\to\infty} S_{\rho l} = I.$$

This limit is independent of the taken sequence ρ^l satisfying condition (3), since if we assume that for a certain sequence $\rho^{l'}$ there takes place

$$\lim_{l'\to\infty} S_{\rho l'} = I_1 \neq I,$$

then the sequence of partitions

$$\rho^1, \rho^{1'}, \rho^2, \rho^{2'}, \rho^3, \ldots$$

would satisfy condition (3), and the corresponding sequence of integral sums

$$S_{\rho 1}, S_{\rho 1'}, S_{\rho 2}, S_{\rho 2'}, S_{\rho 3}, \ldots$$

would not tend to the limit, which contradicts inequality (2) that was already proved.

If $m\Omega = 0$, then the integral of f over Ω, obivously, exists and is equal to zero.

Sec. 2.6. Change of Variables in Multiple Integral. Simplest Case

Let us show how the integral

$$\int\int_{\Omega'} f(x_1', x_2')\, dx_1'\, dx_2' \tag{1}$$

is modified due to the change of variables:

$$\begin{aligned} x_1' &= ax_1 + bx_2, \\ x_2' &= cx_1 + dx_2 \end{aligned} \quad \left(D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0\right). \tag{2}$$

Let us suppose that Ω' is a domain with a continuous piecewise smooth boundary curve Γ' (Fig. 38).

The transformation inverse to (2) specifies a mapping of Ω' onto a domain Ω in the x_1x_2-plane having a piecewise smooth boundary Γ (Fig. 39) and determines the function

$$F(x_1, x_2) = f(ax_1 + bx_2, cx_1 + dx_2), \quad (x_1, x_2) \in \Omega$$

defined on Ω.

Let us construct a rectangular network in the x_1x_2-plane with squares Δ having sides of length h. Equations (2) determine a transformation of this network into a network lying

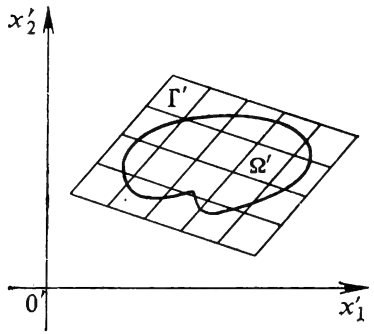

Fig. 38.

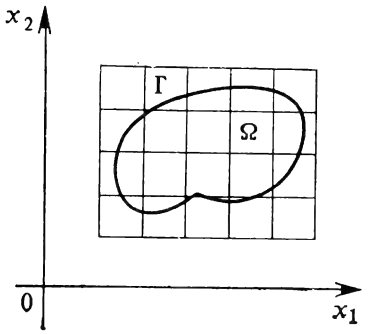

Fig. 39.

in the x_1x_2-plane which is oblique-angled in the general case and partitions the $x_1'x_2'$-plane into congruent parallelograms Δ' (the images of Δ's) with areas (the appropriate explanations are given below)

$$|\Delta'| = |D| \cdot |\Delta| = h^2 |D|. \tag{3}$$

Thus we obtain some partitions ρ and ρ' of the domains Ω and Ω', respectively, specified by these networks.

Let us explain equality (3) in more detail. An arbitrary square Δ is defined by two vectors $(h, 0)$ and $(0, h)$ which will be regarded as emanating from the left-side lower vertex of Δ. The first vector (directed line segment) coincides with the base of Δ and the second with the vertical side of Δ. Transformation (2) maps these vectors into vectors (ah, ch) and (bh, dh), respectively, the latter being sides of the

parallelogram Δ'. The area of Δ', as we know,* is equal to

$$|\Delta'| = \begin{Vmatrix} ah & ch \\ bh & dh \end{Vmatrix} = h^2 |D|.$$

We have

$$S_{\rho'}(f) = \sum f(x_1', x_2') |\Delta'| = \sum F(x_1, x_2) |D| \cdot |\Delta| = S_\rho(|D| F) \quad (4)$$

$$((x_1, x_2) \in \Delta, \quad (x_1', x_2') \in \Delta').$$

We extend the second sum in (4) only over the entire squares $\Delta \subset \Omega$ and the first sum only over the entire parallelograms Δ' corresponding to them (see Sec. 2.1, Remark 4). On passing to the limit in (4) as $h \to 0$, we arrive at the formula

$$\iint\limits_{\Omega'} f(x_1', x_2')\, dx_1'\, dx_2' = \iint\limits_{\Omega} F(x_1, x_2) |D|\, dx_1\, dx_2 = |D| \iint\limits_{\Omega} F(x_1, x_2)\, dx_1\, dx_2. \quad (5)$$

In this argument we may assume that the function f is continuous on $\bar{\Omega}'$, and then the function F is continuous on $\bar{\Omega}$. But this result remains true also in the case when f is bounded on Ω' and continuous everywhere on Ω' excepting individual points or piecewise smooth lines.

In the next section we shall derive and prove a more general formula for change of variables in multiple integrals.

Sec. 2.7. Change of Variables in Multiple Integral. General Case

Let us first confine ourselves to the two-dimensional case. Here we take two functions

$$\left.\begin{aligned} x_1' &= \varphi(x_1, x_2), \\ x_2' &= \psi(x_1, x_2), \end{aligned}\right\} (x_1, x_2) \in \Omega, \quad (1)$$

* See our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 12.

having continuous derivatives on the closure $\bar{\Omega}$ of a bounded domain with a piecewise smooth boundary. We shall assume that transformation (1) maps Ω one-to-one onto a certain domain Ω' with a piecewise smooth boundary.

Let us be given a function $f(\mathbf{x}')$ continuous on $\bar{\Omega}'$ or bounded on Ω' and continuous everywhere except for individual points and piecewise smooth lines.

Under these conditions formula (5) (see the preceding section) for changing variables in a multiple integral is retained, but this time the role of the determinant D is played by the Jacobian*

$$D(x) = \frac{D(x_1', x_2')}{D(x_1, x_2)} = \begin{vmatrix} \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial x_2} \\ \frac{\partial \psi}{\partial x_1} & \frac{\partial \psi}{\partial x_2} \end{vmatrix} \neq 0:$$

$$\iint\limits_{\Omega'} f(x_1', x_2')\, dx_1'\, dx_2' = \iint\limits_{\Omega} f[\varphi(x_1, x_2), \psi(x_1, x_2)] \mid D(x) \mid dx_1\, dx_2. \quad (2)$$

The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, mentioned in Sec. 2.6, may also be regarded as the Jacobian of the linear system

$$x_1' = ax_1 + bx_2, \quad x_2' = cx_1 + dx_2.$$

In the three-dimensional case the formula for change of variables in a multiple integral looks as follows:

$$\iiint\limits_{\Omega'} f(x_1', x_2', x_3')\, dx_1'\, dx_2'\, dx_3' = \iiint\limits_{\Omega} f[\varphi(x_1, x_2, x_3), \psi(x_1, x_2, x_3), \chi(x_1, x_2, x_3)] \times \mid D(x) \mid dx_1\, dx_2\, dx_3, \quad (3)$$

where

$$\left.\begin{aligned} x_1' &= \varphi(x_1, x_2, x_3), \\ x_2' &= \psi(x_1, x_2, x_3), \\ x_3' &= \chi(x_1, x_2, x_3), \end{aligned}\right\} \quad (x_1, x_2, x_3) \in \Omega, \quad (4)$$

* Jacobi, Karl Gustav Jacob (1804-1851). German algebraist and analyst.

are continuously differentiable functions on the closure $\bar{\Omega}$ of the domain Ω with a piecewise smooth boundary and

$$D(\mathbf{x}) = \begin{vmatrix} \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial x_2} & \frac{\partial \varphi}{\partial x_3} \\ \frac{\partial \psi}{\partial x_1} & \frac{\partial \psi}{\partial x_2} & \frac{\partial \psi}{\partial x_3} \\ \frac{\partial \chi}{\partial x_1} & \frac{\partial \chi}{\partial x_2} & \frac{\partial \chi}{\partial x_3} \end{vmatrix}.$$

Here we suppose that the domain Ω is mapped one-to-one onto Ω' with the aid of (4). As before, we assume that $f(\mathbf{x}')$ is continuous on $\bar{\Omega}'$ or bounded and continuous on Ω' with the exception of a finite number of points, piecewise smooth lines and piecewise smooth surfaces.

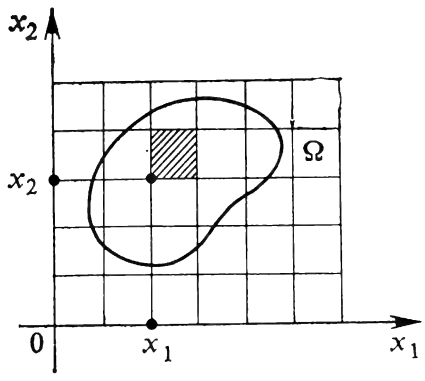

Fig. 40.

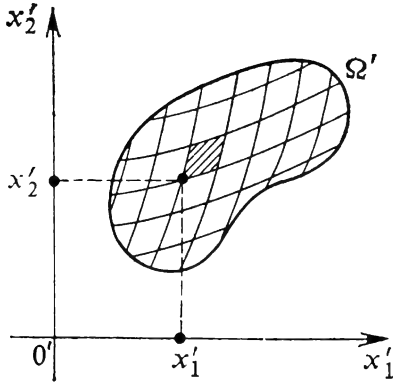

Fig. 41.

Figures 40 and 41 represent the domains Ω and Ω' in the x_1x_2- and $x_1'x_2'$-planes, respectively. The rectangular network dividing the x_1x_2-plane into squares Δ with side of length h goes into an oblique network which partitions Ω' into curvilinear parallelograms Δ'.

Let us consider an arbitrary square $\Delta \subset \Omega$ (Fig. 42). By means of transformations (1), it is mapped onto a curvilinear parallelogram Δ' (Fig. 43). From the vertex of Δ' having coordinates (x_1', x_2') the following vectors are emanated tangent to the "sides" of Δ':

$$\left(h\frac{\partial \varphi}{\partial x_1}, \; h\frac{\partial \psi}{\partial x_1}\right), \quad \left(h\frac{\partial \varphi}{\partial x_2}, \; h\frac{\partial \psi}{\partial x_2}\right)$$

(with x_2 fixed, the slope of the tangent to the curve $\begin{cases} x_1' = \varphi(x_1, x_2), \\ x_2' = \psi(x_1, x_2) \end{cases}$ is equal to $\frac{\partial\psi}{\partial x_1} \Big/ \frac{\partial\varphi}{\partial x_1}$). These vectors replace the respective "sides" with an accuracy up to infinitesimals of higher order (as $h \to 0$!). The parallelogram Δ'' constructed on these vectors has an exactly computable area

$$|\Delta''| = h^2 \left\| \begin{matrix} \frac{\partial\varphi}{\partial x_1} & \frac{\partial\psi}{\partial x_1} \\ \frac{\partial\varphi}{\partial x_2} & \frac{\partial\psi}{\partial x_2} \end{matrix} \right\| = h^2 \left| \frac{D(x_1', x_2')}{D(x_1, x_1)} \right| = h^2 |D(\mathbf{x})|.$$

We can accurately show that the area (two-dimensional measure) of the curvilinear parallelogram Δ' has the form

$$|\Delta'| = (|D(\mathbf{x})| + \varepsilon_\Delta) h^2,$$

where the quantity ε_Δ tends to zero, as $h \to 0$, uniformly for all the squares Δ. For each square Δ the quantity ε_Δ

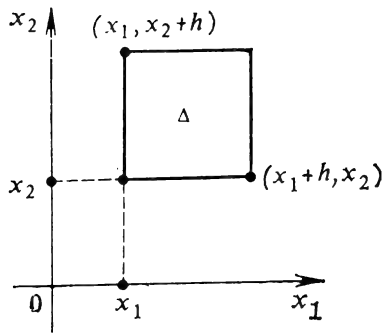

Fig. 42.

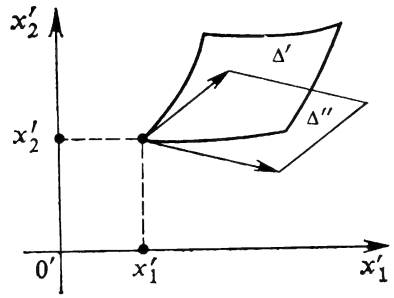

Fig. 43.

depends on Δ and tends to zero as $h \to 0$. The uniformity with which it tends to zero manifests itself in that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|\varepsilon_\Delta| < \varepsilon, \ \forall\, \Delta \ \text{with} \ h < \delta.$$

Let us consider the integral sum of the function $f(x')$ corresponding to the partition Ω' shown in Fig. 41. Here we extend the sum over the "entire" Δ', i.e. such that correspond to the squares Δ entirely belonging to Ω ($\Delta \subset \Omega$).

Then

$$\sum{}' f(\mathbf{x}') \mid \Delta' \mid = \sum{}' f[\varphi(\mathbf{x}), \psi(\mathbf{x})] [\mid D(\mathbf{x}) \mid + \varepsilon_\Delta] h^2$$

$$= \sum{}' f[\varphi(\mathbf{x}),$$

$$\psi(\mathbf{x})] \mid D(\mathbf{x}) \mid h^2 + r_n \xrightarrow[h \to 0]{} \int_\Omega f[\varphi(\mathbf{x}), \psi(\mathbf{x})] \mid D(\mathbf{x}) \mid d\mathbf{x}.$$

Here $\mathbf{x}'$ is an arbitrary point belonging to Δ' ($\mathbf{x}' \in \Delta'$), and $\mathbf{x}$ is the point corresponding to $\mathbf{x}'$ with the aid of (1), obviously, belonging to Δ ($\mathbf{x} \in \Delta$). $\sum'$ symbolizes that the sum is extended only to the entire squares of Δ ($\Delta \subset \Omega$). Further

$$\mid r_n \mid = \left| \sum{}' f[\varphi(\mathbf{x}), \psi(\mathbf{x})] \varepsilon_\Delta h^2 \right| \leqslant M\varepsilon \sum{}' h^2 \leqslant M \cdot \varepsilon m\Omega \times (M \geqslant \mid f[\varphi(\mathbf{x}), \psi(\mathbf{x})] \mid),$$

whence it is seen that

$$r_n \xrightarrow[h \to 0]{} 0.$$

Formula (2) has been proved.

Sec. 2.8. System of Polar Coordinates in the Plane

A ray emanating from a given point O is called the *polar axis* and the point O is referred to as the *pole* of polar coordinates (Fig. 44). An arbitrary point A in the plane has polar

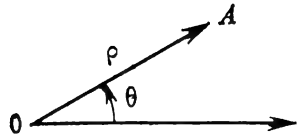

Fig. 44.

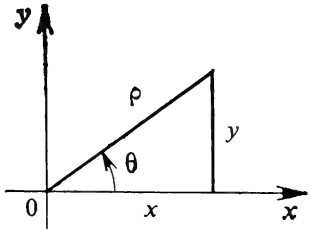

Fig. 45.

coordinates (ρ, θ), where ρ is the distance from A to O, and θ is the angle between the vector (directed line-segment) $\overrightarrow{OA}$ and the polar axis measured from the latter anticlockwise.

Let us introduce a system of rectangular coordinates x, y whose positive axis x coincides with the polar axis (Fig. 45).

The system of equations

$$x = \rho \cos \theta, \quad y = \rho \sin \theta \tag{1}$$

transforms polar coordinates into Cartesian (rectangular) ones. The right-hand members in equalities (1) are continuously differentiable functions with the Jacobian

$$\frac{D(x, y)}{D(\rho, \theta)} = \begin{vmatrix} \cos \theta & -\rho \sin \theta \\ \sin \theta & \rho \cos \theta \end{vmatrix} = \rho \geqslant 0. \tag{2}$$

The equation

$$\rho = \psi(\theta) \quad (\theta_1 \leqslant \theta \leqslant \theta_2),$$

where $\psi(\theta)$ is a function, continuous on the closed interval $[\theta_1, \theta_2]$, defining in polar coordinates the curve Γ, i.e. a locus of points whose polar coordinates satisfy this equation.

We shall suppose that $0 < \theta_2 - \theta_1 \leqslant 2\pi$. Then the curve Γ is such that any ray emanating from the pole O at an angle θ to the x-axis, where $\theta_1 < \theta < \theta_2$, intersects Γ at one point (Fig. 46).

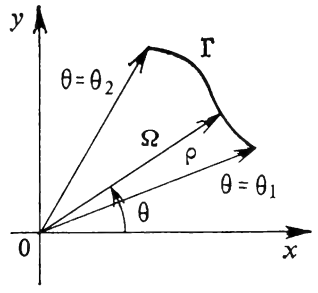

Fig. 46.

Let there be given in the xy-plane a domain Ω bounded by two rays $\theta = \theta_1$, $\theta = \theta_2$ and a curve Γ. Under the aforesaid conditions any point $(x, y) \in \Omega$ corresponds (with the aid of equations (1)) only to one pair (ρ, θ), where $\theta_1 < \theta < \theta_2$. Let now on the closure $\bar{\Omega}$ of our domain there be given a continuous function $f(x, y)$ of (x, y) or it may be bounded on Ω and continuous everywhere excepting individual points and smooth lines. Then the following equality takes place

$$\iint_{\Omega} f(x, y)\, dx\, dy = \int_{\theta_1}^{\theta_2} d\theta \int_{0}^{\psi(\theta)} f(\rho \cos \theta, \rho \sin \theta)\, \rho\, d\rho. \tag{3}$$

According to formula (2) from the preceding section, we have replaced x, y by ρ, θ using equalities (1) and introduced as a factor the modulus of the Jacobian $|\rho| = \rho$. For

the domain Ω' of pairs (ρ, θ), corresponding to the initial domain Ω, the limits are specified at once: we first integrate with respect to ρ from 0 to $\psi(\theta)$, and then with respect to θ from θ_1 to θ_2.

Example 1.

$$\iint\limits_{x^2+y^2\leqslant R^2} \exp(x^2+y^2)\,dx\,dy = \int\limits_0^{2\pi} d\theta \int\limits_0^R \exp\rho^2 \cdot \rho\, d\rho$$

$$= \pi \int\limits_0^R \exp\rho^2 d(\rho^2) = (u=\rho^2) = \pi \int\limits_0^{R^2} e^u\, du = \pi\,[e^{R^2}-1].$$

We have followed formula (3). It should be taken into consideration that the domain defined in Cartesian coordinates

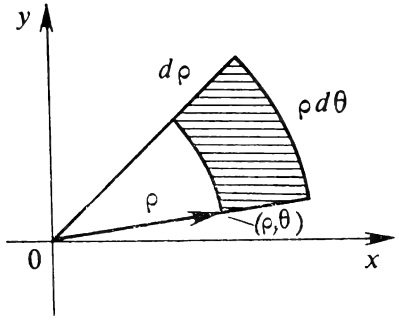

Fig. 47.

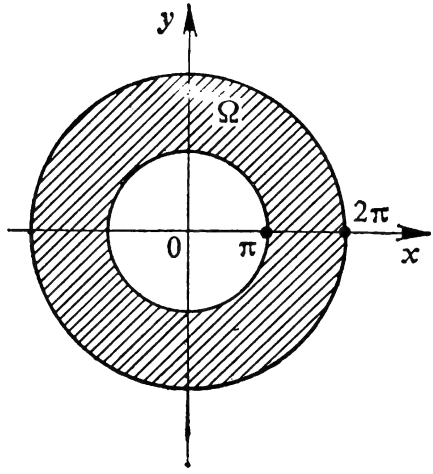

Fig. 48.

by the inequality $x^2 + y^2 < R^2$ is determined in polar coordinates by the inequality $\rho < R$.

Formula (3) can be obtained from natural arguments without resorting to the general formula (2) derived in the preceding section.

We divide the xy-plane into elementary figures by means of closely drawn concentric circles and rays emanating from the pole of the polar system (Fig. 47). The area of an arbitrary elementary figure (located at the point (ρ, θ)) or, as we say, the element of area in polar coordinates, is equal with

an accuracy up to infinitesimals of a higher order to $\Delta s \sim$ $\sim \rho\, d\rho\, d\theta$ (the figure hatched in Fig. 47 may be approximately taken for a rectangle with sides $d\rho$ and $\rho\, d\theta$). Therefore, summating with respect to these elements, we obtain

$$\lim_{\Delta\rho,\ \Delta\theta\to 0} \sum F_j \rho_j \Delta\rho_j \Delta\theta_j = \iint\limits_{\Omega'} F(\rho, \theta)\, \rho\, d\rho\, d\theta,$$

where

$$F(\rho, \theta) = f(\rho \cos\theta, \rho \sin\theta).$$

Example 2. Compute the integral

$$I = \iint\limits_{\pi^2 < x^2 + y^2 \leqslant 4\pi^2} \sin \sqrt{x^2 + y^2}\, dx\, dy.$$

Passing to polar coordinates (Fig. 48), we get

$$I = \int_0^{2\pi} \int_{\pi}^{2\pi} \sin\rho \cdot \rho\, d\rho\, d\theta = 2\pi \int_{\pi}^{2\pi} \rho \sin\rho\, d\rho$$

$$= (\rho = u,\ \sin\rho\, d\rho = dv)$$

$$= 2\pi \left[-\rho\cos\rho \Big|_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos\rho\, d\rho \right]$$

$$= -6\pi^2 + 2\pi \sin\rho \Big|_{\pi}^{2\pi} = -6\pi^2.$$

Sec. 2.9. System of Polar Coordinates in Space

The system of equations

$$x = \rho\cos\theta\cos\varphi, \quad y = \rho\cos\theta\sin\varphi, \quad z = \rho\sin\theta \qquad (1)$$

enables us to pass from polar (spherical) coordinates in space to rectangular coordinates (Fig. 49). Here ρ is the distance of the point $P(x, y, z)$ to the origin (the pole of the polar system), θ is the angle between the radius vector $\underline{\rho}$ of the point P and its projection on the xy-plane, and φ is the angle between the mentioned projection and the positive direction of the x-axis. The angle φ is measured in the direction in which the x-axis should be rotated about the

z-axis to reach the y-axis in the shortest way. We may hold that $0 \leqslant \varphi < 2\pi$ and $-\pi/2 \leqslant \theta \leqslant \pi/2$.

The functions on the right in (1) are continuously differentiable with the Jacobian

$$D = \frac{D(x, y, z)}{D(\rho, \varphi, \theta)} = \begin{vmatrix} \cos\theta\cos\varphi & -\rho\cos\theta\sin\varphi & -\rho\sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta & 0 & \rho\cos\theta \end{vmatrix} = \rho^2\cos\theta. \tag{2}$$

Let σ be a surface described in polar coordinates by the function $\rho = \psi(\theta, \varphi)$ $((\theta, \varphi) \in \omega)$, continuous on the closure

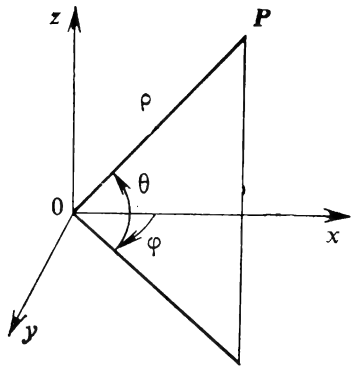

Fig. 49.

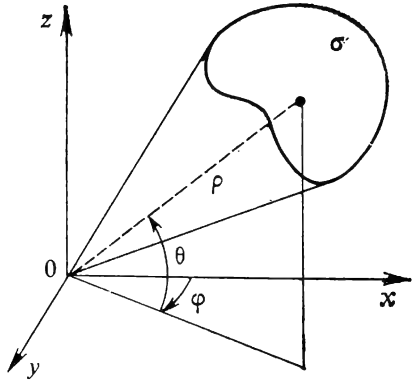

Fig. 50.

of the domain ω and let Ω be a three-dimensional domain of the xyz-space bounded by the surface σ and a conical surface whose rays emanate from the origin to intersect the boundary of σ (see Fig. 50). Then for a function $f(x, y, z)$, continuous on $\bar{\Omega}$, there takes place the following equality

$$\iiint_{\Omega} f(x, y, z)\, dx\, dy\, dz = \iint_{\omega} d\theta\, d\varphi \int_0^{\psi(\theta, \varphi)} F(\rho, \theta, \varphi)\, \rho^2 \cos\theta\, d\rho, \tag{3}$$

where $F(\rho, \theta, \varphi) = f(\rho \cos\theta \cos\varphi, \rho \cos\theta \sin\varphi, \rho \sin\theta)$. We have taken advantage of the general formula (3) from Sec. 2.7, taking into account equalities (1) and (2). In the present case $-\pi/2 \leqslant \theta \leqslant \pi/2$, therefore $\rho^2 \cos\theta \geqslant 0$.

To visually obtain an element of volume in polar coordinates, let us cut the space into small parts by means of

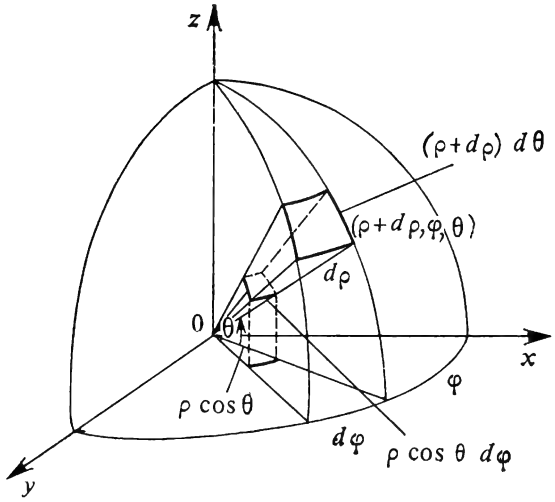

Fig. 51.

concentric spherical surfaces with centre at the pole (point $\rho = 0$), planes passing through the z-axis, and conical surfaces determined by angles θ and $\theta + d\theta$ (Fig. 51), having the z-axis as their axis. It is easy to see that the small cells thus obtained may be approximately regarded as rectangular parallelepipeds with edges $\rho\, d\theta$, $d\rho$, $\rho \cos\theta\, d\varphi$, therefore their volumes can be computed to within infinitesimals of a higher order by the formula

$$\Delta v = \rho^2 \cos\theta\, d\rho\, d\theta\, d\varphi,$$

where (ρ, θ, φ) is one of the points of a cell.

Example. Compute the triple integral

$$I = \iiint_{\Omega} xyz\, dx\, dy\, dz,$$

where Ω is a domain of points with positive coordinates bounded by the surfaces $x = 0$, $y = 0$, $z = 0$, and $x^2 + y^2 + z^2 = 1$.

Let us introduce polar (spherical) coordinates using formulas (1), then for the domain Ω: $0 \leqslant \rho \leqslant 1$, $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant \varphi \leqslant \pi/2$. According to formula (3), we have

$$I = \int_0^{\pi/2} d\varphi \int_0^{\pi/2} d\theta \int_0^1 \rho^5 \cos^3\theta \sin\theta \cos\varphi \sin\varphi \, d\rho$$

$$= \int_0^{\pi/2} -\cos\varphi \, d\cos\varphi \int_0^{\pi/2} -\cos^3\theta \, d\cos\theta \int_0^1 \rho^5 \, d\rho$$

$$= -\frac{\cos^2\varphi}{2}\Big|_0^{\pi/2} \left(-\frac{\cos^4\theta}{4}\right)\Big|_0^{\pi/2} \frac{\rho^6}{6}\Big|_0^1 = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6} = \frac{1}{48}.$$

Sec. 2.10. Cylindrical Coordinates

Let there be given a system of rectangular coordinates x, y, z in a three-dimensional space. An arbitrary point $A = (x, y, z)$ of this space is also described by a triple of numbers (ρ, θ, z), where z is, as before, its z-coordinate, and (ρ, θ) are polar coordinates of a point (x, y) of the xy-plane in the supposition that the polar axis coincides with the positive direction of the x-axis (Fig. 52). Obviously,

$$\left.\begin{aligned} x &= \rho\cos\theta, \\ y &= \rho\sin\theta, \\ z &= z. \end{aligned}\right\} \qquad (1)$$

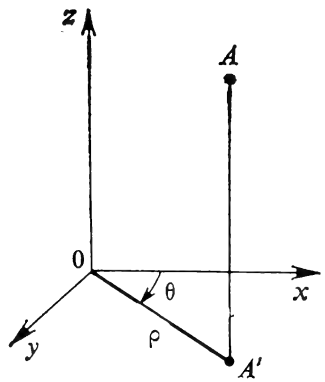

Fig. 52.

The Jacobian of this transformation

$$D = \frac{D(x, y, z)}{D(\rho, \theta, z)} = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho \geqslant 0. \qquad (2)$$

In this case the formula for change of variables is written as follows:

$$\iiint\limits_{\Omega} f(x, y, z)dx\,dy\,dz = \iiint\limits_{\Omega'} f[\rho\cos\theta,\ \rho\sin\theta,\ z]\,\rho\,d\rho\,d\theta\,dz.$$

To obtain visually an element of volume in cylindrical coordinates, let us cut the space by concentric circular cylindrical surfaces with the z-axis as their axis, planes

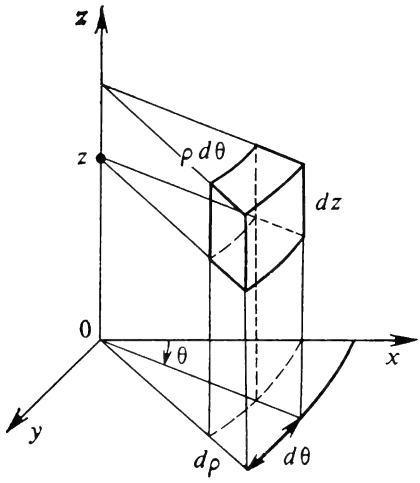

Fig. 53.

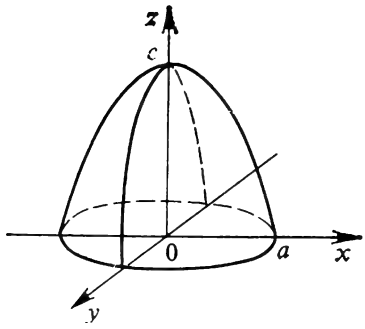

Fig. 54.

passing through the z-axis, and planes parallel to the xy-plane (Fig. 53). The element of our space bounded by these surfaces, with an accuracy to infinitesimals of higher order, represents a rectangular parallelepiped with edges $d\rho$, dz, $\rho\,d\theta$. Its volume is equal to $\rho\,d\rho\,d\theta\,dz$.

Example. Find the volume of the body Ω bounded by the surfaces

$$\left(\frac{x^2+y^2}{a^2}\right)^2+\frac{z}{c}=1,\quad z=0,\quad a,\ c>0 \text{ (Fig. 54)}.$$

As we know

$$V=m\Omega=\iiint\limits_{\Omega} dx\,dy\,dz.$$

Introducing cylindrical coordinates (1), we obtain

$$V = \iiint_{\Omega'} \rho \, d\rho \, d\theta \, dz = \int_0^a \int_0^{2\pi} \rho \, d\rho \, d\theta \int_0^{c\left[1-\frac{\rho^4}{a^4}\right]} dz$$

$$= 2\pi c \int_0^a \rho \left(1 - \frac{\rho^4}{a^4}\right) d\rho = \frac{2}{3} \pi a^2 c.$$

Sec. 2.11. Surface Area

Let us consider the three-dimensional space $R = R_3$ with rectangular coordinates (x, y, z) and a surface S described by an equation

$$z = f(x, y) \quad ((x, y) \in \bar{G}). \tag{1}$$

We suppose that G is an open bounded domain of the xy-plane with a piecewise smooth boundary, and that the function f possesses the continuous partial derivatives

$$p = \frac{\partial f}{\partial x}, \quad q = \frac{\partial f}{\partial y} \tag{2}$$

in $\bar{G}$. Let us cut $\bar{G}$ into parts any two of which either do not intersect or intersect only along their (piecewise smooth) boundaries

$$\bar{G} = \sum_{j=1}^{N} G_j.$$

Let (x_j, y_j) be an arbitrary point G_j $(j = 1, \ldots, N)$. To this point there corresponds the point $P_j \in S$ with coordinates (x_j, y_j, f_j) where $f_j = f(x_j, y_j)$. Let us draw through the point P_j the tangent plane L_j to S. The cylindrical surface Γ_j, with the boundary of G_j as directrix and generatrix (elements) parallel to the z-axis, cuts a portion l_j out of the tangent plane L_j. Let $|\, l_j \,|$ denote the area of that portion.

The *area of the surface* S is defined as the limit

$$|\, S \,| = \lim_{\max d(G_j) \to 0} \sum_{j=1}^{N} |\, l_j \,|.$$

The cosine of the acute angle between the normal $\mathbf{n}_j$ to S at the point P_j and the z-axis is equal to

$$\cos(\mathbf{n}_j, z) = 1/\sqrt{1+p_j^2+q_j^2},$$

where the square root is taken with the plus sign, and p_j and q_j are the results of the substitution of x_j and y_j into p and q. We know that the equation of a tangent plane has the form*

$$(Z - z_j) - p_j(X - x_j) - q_j(Y - y_j) = 0$$

and, consequently, a normal to it is determined by the vector $(-p_j, -q_j, 1)$. G_j is obviously the projection of l_j on the xy-plane and, hence, the area (two-dimensional measure) of G_j is equal to

$$|G_j| = |l_j| \cos(\mathbf{n}_j, z),$$
$$|l_j| = |G_j| \sqrt{1+p_j^2+q_j^2}$$

and

$$|S| = \lim_{\max d(G_j)\to 0} \sum_{j=1}^{N} |l_j|$$
$$= \lim_{\max d(G_j)\to 0} \sum_{j=1}^{N} \sqrt{1+p_j^2+q_j^2}\,|G_j|$$
$$= \iint_G \sqrt{1+p^2+q^2}\,dx\,dy.$$

since the function $\sqrt{1+p^2+q^2}$ is continuous on $\overline{G}$, and, consequently, is integrable.

We have proved that the area of the surface S described by equation (1) is expressed by the formula

$$|S| = \iint_G \sqrt{1+(f'_x)^2+(f'_y)^2}\,dx\,dy. \qquad (3)$$

Of course, if the surface S is defined with the aid of the function

$$x = F(y, z) \quad ((y, z) \in H), \qquad (1')$$

* See our book *Higher Mathematics. Differential and Integral Calculus*, Secs. 8.7, 8.8.

expressing explicitly the dependence of x on y, z then

$$|S| = \iint\limits_H \sqrt{1 + (F'_y)^2 + (F'_z)^2}\, dy\, dz, \qquad (3')$$

and if S is given by the function

$$y = \Phi(x, z) \quad ((x, z) \in \Lambda), \qquad (1'')$$

then

$$|S| = \iint\limits_\Lambda \sqrt{1 + (\Phi'_x)^2 + (\Phi'_z)^2}\, dx\, dz. \qquad (3'')$$

Example 1. Compute the area of the part of sphere $x^2 + y^2 + z^2 = a^2$ enclosed inside a circular cylinder $x^2 + y^2 = b^2$ $(b \leqslant a)$.

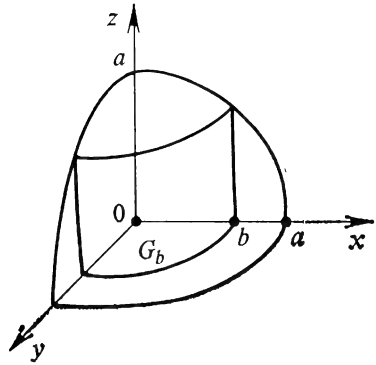

Fig. 55.

In this case $z = \pm\sqrt{a^2 - x^2 - y^2}$. Let G_b be a quarter of the circle of radius b with centre at the origin (Fig. 55). According to (3), taking into account the symmetry, we find

$$|S_b| = 8 \iint\limits_{G_b} \sqrt{1 + \left(\frac{x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}$$

$$\times\, dx\, dy = 8 \iint\limits_{G_b} \frac{a\, dx\, dy}{\sqrt{a^2 - x^2 - y^2}} = 8a \int\limits_0^b \int\limits_0^{\pi/2} \frac{\rho\, d\rho\, d\theta}{\sqrt{a^2 - \rho^2}}$$

$$= 2\pi a \left[-2\,(a^2 - \rho^2)^{1/2}\,\Big|_0^b\right] = 4\pi a\left(a - \sqrt{a^2 - b^2}\right).$$

For $b = a$ the integrals in this chain should be understood in an improper sense (see below Sec. 2.13).

Let us transform integral (3) with the aid of the substitution

$$x = \varphi(u, v), \quad y = \psi(u, v) \quad ((u, v) \in \bar{\Omega}), \tag{4}$$

which establishes one-to-one correspondence between the domains Ω and G, in the supposition that φ and ψ are continuously differentiable on $\bar{\Omega}$ and the Jacobian

$$\frac{D(x, y)}{D(u, v)} \neq 0 \text{ on } \Omega. \tag{5}$$

Substituting the expressions for x and y from (4), we obtain

$$z = f[\varphi(u, v), \psi(u, v)] = \chi(u, v).$$

Then

$$\frac{\partial \chi}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial \psi}{\partial u},$$
$$\frac{\partial \chi}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial \psi}{\partial v}.$$

Solving these equations with respect to $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$ and taking into consideration (5), we get

$$p = -\frac{D(y, z)}{D(u, v)} \Big/ \frac{D(x, y)}{D(u, v)}, \quad q = -\frac{D(z, x)}{D(u, v)} \Big/ \frac{D(x, y)}{D(u, v)}.$$

Therefore, applying to the integral (3) the formula for a change of variables (2) from Sec 2.7, we obtain

$$|S| = \iint_G \sqrt{1 + p^2 + q^2}\, dx\, dy$$

$$= \iint_\Omega \sqrt{1 + \left(\frac{D(y, z)}{D(u, v)} \Big/ \frac{D(x, y)}{D(u, v)}\right)^2 + \left(\frac{D(z, x)}{D(u, v)} \Big/ \frac{D(x, y)}{D(u, v)}\right)^2} \times \left|\frac{D(x, y)}{D(u, v)}\right| du\, dv. \tag{6}$$

Thus, the area of the surface S is also expressed by the formula

$$|S| = \iint_\Omega \sqrt{\left(\frac{D(x, y)}{D(u, v)}\right)^2 + \left(\frac{D(y, z)}{D(u, v)}\right)^2 + \left(\frac{D(z, x)}{D(u, v)}\right)^2} \times du\, dv. \tag{7}$$

A smooth surface S can be represented parametrically with the aid of three equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in \Omega, \tag{8}$$

where the functions x, y, z have continuous partial derivatives on the closure $\overline{\Omega}$ of the domain Ω of values (u, v) called the parameters of S. It is supposed here that the rank of the matrix

$$\begin{Vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{Vmatrix} \tag{9}$$

is equal to 2 for all the points $(u, v) \in \Omega$. This shows that there takes place the inequality

$$\left(\frac{D(x, y)}{D(u, v)}\right)^2 + \left(\frac{D(y, z)}{D(u, v)}\right)^2 + \left(\frac{D(z, x)}{D(u, v)}\right)^2 > 0, \quad \forall (u, v) \in \Omega. \tag{10}$$

Indeed, this inequality expresses the fact that for any $(u, v) \in \Omega$ at least one of the terms of the sum on the left-hand side of (10) is non-zero which is equivalent to that one of the second-order determinants generated by matrix (9) is not equal to zero.

Note that three equations (8) of the surface S can be written in the vector form:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \tag{11}$$

where the vector $\mathbf{r} = \mathbf{r}(u, v)$ (the radius vector of a point on the surface S) is thus dependent on two scalar parameters u and v.

Differentiating equality (11) with respect to u and v, we get

$$\left.\begin{aligned} \mathbf{r}_u &= \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \\ \mathbf{r}_v &= \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}. \end{aligned}\right\} \tag{12}$$

The vector product of these two vectors is

$$\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u} \right) \mathbf{i}$$

$$+ \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u} \right) \mathbf{j} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \mathbf{k}$$

$$= \mathbf{i}\frac{D(y, z)}{D(u, v)} + \mathbf{j}\frac{D(z, x)}{D(u, v)} + \mathbf{k}\frac{D(x, y)}{D(u, v)}, \qquad (13)$$

i.e. the squared length of the vector $\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v$ turns out to be equal to the left-hand side of (10):

$$|\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v|^2 = \left(\frac{D(x, y)}{D(u, v)} \right)^2 + \left(\frac{D(y, z)}{D(u, v)} \right)^2 + \left(\frac{D(z, x)}{D(u, v)} \right)^2.$$

The equations

$$x = \cos\theta \cos\varphi, \quad y = \cos\theta \sin\varphi, \quad z = \sin\theta$$

define a spherical surface of radius 1 with centre at the zero point ($x^2 + y^2 + z^2 = 1$). We see that the entire spherical surface can be represented in the parametric form by single equations. Obviously, it is impossible to define a spherical surface as a whole explicitly, i.e. in one of the forms (1), (1′), (1″). Only separate pieces of a spherical surface can be described explicitly if this surface is projected identically on this or that coordinate plane.

On the other hand, a surface S represented parametrically with the aid of equations (8) can always be expressed explicitly, but only locally.[5] Indeed, let us take some point $A \in S$ corresponding to the parameters (u_0, v_0). By virtue of condition (10), one of the terms of the left-hand side of (10) is greater than zero, say, the first. But then in a neighbourhood of (u_0, v_0) we can solve the first two equations of (8) with respect to u and v and obtain

$$u = \lambda(x, y), \quad v = \mu(x, y).$$

Substituting these functions into the third equation of (8), we obtain that a certain piece σ of the surface S containing

the point A is explicitly described by an equation of the form

$$z = f(x, y).$$

We see that a parametric representation of a surface is, generally speaking, more general than an explicit one.

If a surface S is represented parametrically by equations (8), then its area is defined as the number equal to

$$|S| = \iint\limits_{\Omega} |\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v| \, du \, dv. \tag{14}$$

Note that to any domain ω belonging to Ω there corresponds (with the aid of equations (8)) some surface $\sigma \subset S$. This is a piece of the surface S. According to (14), the area of σ is equal to

$$|\sigma| = \iint\limits_{\omega} |\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v| \, du \, dv. \tag{15}$$

It is evident that if σ is cut into two pieces σ_1, σ_2, corresponding to the domains ω_1, ω_2 ($\overline{\omega} = \overline{\omega_1} + \overline{\omega_2}$), then

$$|\sigma| = \iint\limits_{\omega} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \iint\limits_{\omega_1} |\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v| \, du \, dv + \iint\limits_{\omega_2} |\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v| \, du \, dv = |\sigma_1| + |\sigma_2|.$$

This shows that definition (14) possesses a natural property: the area of the piece σ of the surface S equals to the sum of the areas of the pieces σ_1 and σ_2 into which it is cut.

We should also like to underline that when applied to the piece σ identically projected on this or that coordinate plane, definition (14) of the area $|\sigma|$ is equivalent to the initial definition of the area of such a piece ((3), (3′) or (3″)). This was proved above (see (6)).

Example 2. Find the area of the surface represented parametrically:

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = b\varphi, \quad 0 < r < a,$$
$$0 \leqslant \varphi \leqslant 2\pi.$$

The given surface is called the helicoid. Let us compute the Jacobians:

$$\frac{D(x, y)}{D(r, \varphi)} = r \quad \frac{D(x, z)}{D(r, \varphi)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ 0 & b \end{vmatrix} = b\cos\varphi,$$

$$\frac{D(y, z)}{D(r, \varphi)} = \begin{vmatrix} \sin\varphi & r\cos\varphi \\ 0 & b \end{vmatrix} = b\sin\varphi.$$

Further,

$$\left[\frac{D(x, y)}{D(r, \varphi)}\right]^2 + \left[\frac{D(y, z)}{D(r, \varphi)}\right]^2 + \left[\frac{D(x, z)}{D(r, \varphi)}\right]^2 = r^2 + b^2.$$

Therefore, by formula (7), we have

$$|S| = \iint_{\Omega} \sqrt{r^2+b^2}\, dr\, d\varphi = \int_0^{2\pi}\int_0^{a} \sqrt{r^2+b^2}\, dr\, d\varphi$$

$$= 2\pi \int_0^{a} \sqrt{r^2+b^2}\, dr = 2\pi\left[\frac{r}{2}\sqrt{r^2+b^2} + \frac{b^2}{2}\ln(r + \sqrt{r^2+b^2})\right]_0^{a} = \pi\left[a\sqrt{a^2+b^2} + b^2 \ln\frac{a+\sqrt{a^2+b^2}}{b}\right].$$

Sec. 2.12. The Coordinates of the Centre of Mass

In space with a rectangular coordinate system introduced, let there be given a material point $P = (x_1, x_2, x_3)$ with mass M_P.

The static moment of this point about the plane $x_i = 0$ is defined as the product $M_P \cdot x_i$ and is denoted as follows

$$K_{x_i} = M_P x_i.$$

The static moment about the plane $x_i = 0$ of a finite system of material points $P_j = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)})$ with masses M_j is determined by the equality

$$K_{x_i} = \sum_j M_j x_i^{(j)}.$$

Finally, if a mass is distributed over a set G, then the static moment of the body G with respect to the plane $x_i = 0$

is defined as the integral

$$K_{x_i} = \int\limits_G \rho(P)\, x_i\, dG = \int\limits_G \rho(P)\, x_i\, dx_1\, dx_2\, dx_3,$$

where $\rho(P)$ is the density of mass distribution.

The centre of gravity P^c of the body G has the coordinates (x_1^c, x_2^c, x_3^c), determined by the inequalities

$$x_i^c = \int\limits_G x_i \rho(P)\, dG \Big/ \int\limits_G \rho(P)\, dG \quad (i = 1, 2, 3).$$

In particular, if $n = 2$ and G is a curvilinear trapezoid in the plane (x_1, x_2) bounded from above by the graph of the

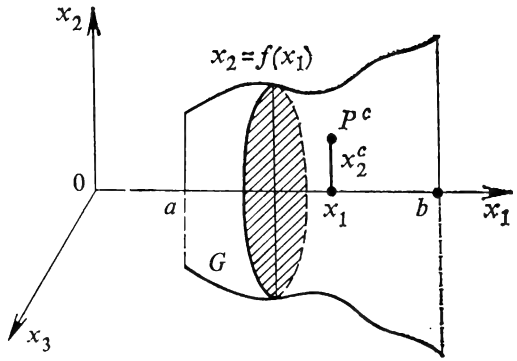

Fig. 56.

function $x_2 = f(x_1)$ and from below by the x_1-axis, containing uniformly distributed masses with density $\rho \equiv 1$, then (Fig. 56)

$$x_2^c = \frac{\int\limits_G x_2\, dx_1\, dx_2}{\int\limits_G dx_1\, dx_2} = \frac{\int\limits_a^b dx_1 \int\limits_0^{f(x_1)} x_2\, dx_2}{mG} = \frac{1}{2mG} \int\limits_a^b f^2(x_1)\, dx_1.$$

Hence,

$$2\pi x_2^c \cdot mG = \pi \int\limits_a^b f^2(x_1)\, dx_1. \qquad (1)$$

The right-hand member of (1) represents the volume of the solid generated by revolving the curvilinear trapezoid G about the x_1-axis.

Thus, we have obtained

Guldin's* (or Pappus') first theorem. *The volume of a solid of revolution is equal to the product of the area of a curvilinear trapezoid G and the circumference of the circle described by the centroid of this trapezoid about the x_1-axis.*

If G is a uniform ($\rho \equiv 1$) curve $x_2 = f(x_1)$, $a \leqslant x_1 \leqslant b$, then

$$x_2^c = \frac{\int\limits_G x_2\, dl}{mG} = \frac{\int\limits_G f(x_1)\, dl}{mG},$$

where mG is the length of the curve within the limits $[a, b]$, and dl is the element of the arc length. Since $dl = \sqrt{1 + f'(x_1)^2}\, dx_1$, then

$$x_2^c = \frac{1}{mG} \int_a^b f(x_1) \sqrt{1 + f'(x_1)^2}\, dx_1$$

or

$$2\pi x_2^c mG = 2\pi \int_a^b f(x_1) \sqrt{1 + f'(x_1)^2}\, dx. \qquad (2)$$

The right-hand member of (2) represents the area of the surface of revolution formed by revolving the curve G ($x_2 = f(x_1)$, $a \leqslant x_1 \leqslant b$) about the x_1-axis.

Thus, equality (2) yields

Guldin's second theorem. *The area of a surface of revolution is equal to the product of the length of the generating curve G ($x_2 = f(x_1)$, $a \leqslant x_1 \leqslant b$) and the circumference of the circle described by the centroid of the curve about the x_1-axis.*

Given two quantities, Guldin's theorems enable us to find the third, unknown, quantity. For instance, if the

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* Guldin, Paul (1577-1643). A Swiss mathematician who formulated a theorem on centroids which had already been explained by Pappus, and who helped in the search for new methods of solving problems on areas, volumes, and centroids.

Pappus of Alexandria (fl. c. 300 A.D.). Greek geometer.

coordinates of the centroid and the volume of a solid of revolution are known, then it is possible to determine the area of a curvilinear trapezoid, and so on.

Example 1. Find the coordinates of the centroid of the curvilinear trapezoid $G = \{-R \leqslant x \leqslant R,\ x^2 \leqslant y \leqslant R^2\}$ shown in Fig. 57.

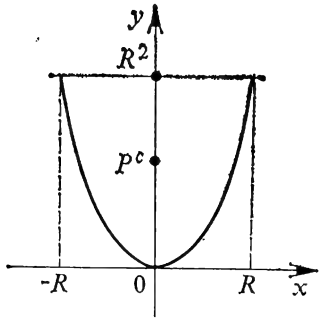

Fig. 57.

Let $P^c = (x^c, y^c)$ be the centroid. By virtue of symmetry, it is clear that $x^c = 0$ (we put $\rho \equiv \equiv 1$). Find the area of the trapezoid G:

$$mG = R^2 \cdot 2R - 2\int_0^R x^2\,dx = $$

$$= 2R^3 - \frac{2}{3}R^3 = \frac{4}{3}R^3.$$

The volume of the solid generated by revolving G about the x-axis is equal to

$$V = 2R \cdot \pi R^4 - 2\pi \int_0^R (x^2)^2\,dx = $$

$$= 2\pi R^5 - \frac{2}{5}\pi R^5 = \frac{8}{5}\pi R^5.$$

By the first Guldin theorem,

$$y^c = \frac{V}{2\pi mG} = \frac{3}{5}R^2.$$

Example 2. Find the volume of the solid generated by revolving the circle $G = \{x^2 + (y - a)^2 \leqslant r^2\}$ of radius r $(r < a)$ with centre at the point $(0, a)$ about the x-axis (Fig. 58).

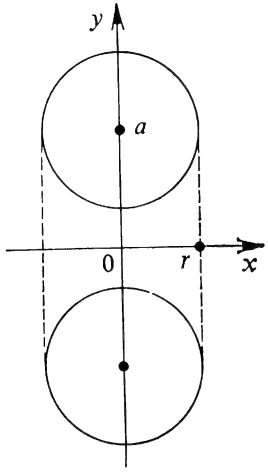

Fig. 58.

It is evident that the centroid of a (homogeneous) disk coincides with its geometrical centre, i.e. $x^c = 0$, $y^c = a$. The area of the disk $mG = \pi r^2$. Therefore, according to the first Guldin theorem

$$V = 2\pi a \cdot \pi r^2 = 2\pi^2 r^2 a.$$

Example 3. Find the surface area of the solid of revolution considered in Example 2.

The given surface may be regarded as a surface generated by revolving the circle $x^2 + (y - a)^2 = r^2$ about the x-axis. The length of the circumference of this circle is equal to $2\pi r$. Therefore, by the second Guldin theorem,

$$S = 2\pi a \cdot 2\pi r = 4\pi^2 ar$$

(the centroid of a homogeneous circle also coincides with the centre $(0, a)$ of this circle).

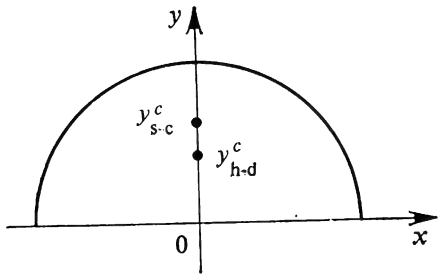

Fig. 59.

Example 4. Find the centroid of a homogeneous ($\rho \equiv 1$) half-disk $x^2 + y^2 \leqslant R^2$, $y > 0$; of a semi-circle $x^2 + y^2 = R^2$, $y > 0$.

As is known, the volume of a ball of radius R is equal to $\frac{4}{3}\pi R^3$, its surface area being equal to $4\pi R^2$. By formula (1), we obtain (Fig. 59).

$$2\pi y^c_{h\text{-}d} \cdot \frac{\pi R^2}{2} = \frac{4}{3}\pi R^3, \quad y^c_{h\text{-}d} = \frac{4R}{3\pi},$$

where $y^c_{h\text{-}d}$ is the ordinate of the centroid of the half-disk.

By formula (2), for the ordinate $y^c_{s\text{-}c}$ of the centroid of the semi-circle we have

$$2\pi y^c_{s\text{-}c} \cdot \pi R = 4\pi R^2, \quad y^c_{s\text{-}c} = \frac{2R}{\pi}.$$

Moments. The moment of the qth order ($q = 2, 3, \ldots$) of a material point P with mass M_P with respect to the plane $x_i = 0$ is defined as the product

$$I^{(q)}_{x_i} = M_P x_i^q.$$

If the masses are distributed over a measurable set G with density $\rho(P)$, then

$$I_{x_i}^{(q)} = \int_G x_i^q \rho(P)\, dG \quad (i = 1, 2, 3).$$

If $q = 2$, then the corresponding moment of the second order is called the *moment of inertia*.

Besides, we can consider the moments of the qth order of the body G about the origin

$$I_0^{(q)} = \int_G \rho(P) \left(\sum_{i=1}^{3} x_i^2 \right)^{q/2} dG;$$

and about the axis x_i $(i = 1, 2, 3)$. For instance, the moment of the qth order about the x_1-axis will be written as

$$I_{x_2x_3}^{(q)} = \int_G \rho(P)\, (x_2^2 + x_3^2)^{q/2}\, dG.$$

Sec. 2.13. Improper Integrals *

Let there be given a function $u = f(x)$ on a closed domain $\bar{G}$ $(\bar{G} = G \cup \partial G)$. The point $x^0 \in G$ is called a *singularity* of the function if in any ε-neighbourhood of the point x^0 the function $f(x)$ is unbounded. Let (Fig. 60) $G_\varepsilon = G \setminus U(x^0, \varepsilon)$, where $U(x^0, \varepsilon)$ is an open ball of radius ε with centre at the point x^0 $(\rho(x, x^0) = |x - x^0| < \varepsilon)$.

If the function $f(x)$ has only one singularity at the point x^0 in the domain $\bar{G}$ and is continuous on the domain G_ε for $\forall\, \varepsilon > 0$, then the *improper integral* of the function $f(x)$ over G is defined as the limit (if the latter exists)

$$\lim_{\varepsilon \to 0} \int_{G_\varepsilon} f(x)\, dx = \int_G f(x)\, dx. \qquad (1)$$

If a finite limit (1) exists, then the *improper integral* is said to be *convergent*, and if limit (1) does not exist or is equal to infinity, then it is said to be *divergent*.

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* In Sec. 2.13 we shall denote the points x, y in space in lightface, but not by letters in bold type.

Integral (1) is called *absolutely convergent* if the integral

$$\int_G |f(x)|dx < \infty \tag{2}$$

of the modulus $|f(x)|$ is convergent.

An absolutely convergent integral converges. Indeed, from the absolute convergence of integral (2) it follows that the

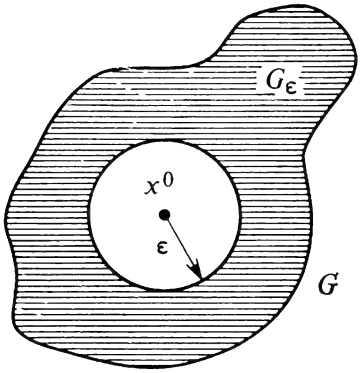

Fig. 60.

limit $\lim\limits_{\varepsilon \to 0} \int\limits_{G_\varepsilon} |f(x)|\,dx$ exists and is finite. But then, applying Cauchy's test (concerning the function of one variable ε) to this limit, we obtain that for any $\eta > 0$ there is $\varepsilon_0 > 0$ such that

$$\eta > \left| \int\limits_{G_\varepsilon} |f|dx - \int\limits_{G_{\varepsilon'}} |f|dx \right| = \int\limits_{G_\varepsilon \setminus G_{\varepsilon'}} |f|dx \geqslant \left| \int\limits_{G_\varepsilon \setminus G_{\varepsilon'}} f\,dx \right|$$

$$\geqslant \left| \int\limits_{G_\varepsilon} f\,dx - \int\limits_{G_{\varepsilon'}} f\,dx \right|, \quad \forall \varepsilon < \varepsilon' < \varepsilon_0.$$

Thus, for any $\eta > 0$ there is ε_0, such that for all positive ε, ε', satisfying the inequalities $0 < \varepsilon < \varepsilon' < \varepsilon_0$, we have

$$\left| \int\limits_{G_\varepsilon} f\,dx - \int\limits_{G_{\varepsilon'}} f\,dx \right| < \eta.$$

And this shows that, according to Cauchy's criterion, limit (1) exists, i.e. improper integral (1) is existent.

Example 1. Investigate for convergence the integral

$$\int\limits_{\Omega} |x|^{-\alpha}\, dx, \tag{3}$$

where $\alpha > 0$, $x = (x_1, x_2, x_3)$, $|x|^2 = x_1^2 + x_2^2 + x_3^2$, Ω is a unit ball $|x| \leqslant 1$.

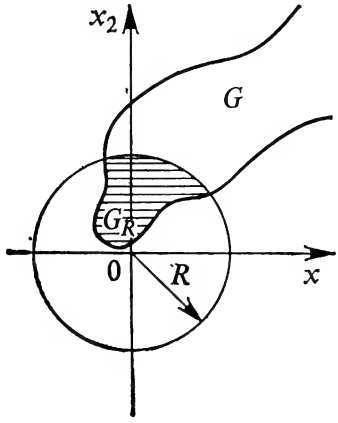

Fig. 61.

Solution. The function $|x|^{-\alpha}$ has only one singularity at the point (0, 0, 0). Therefore, passing to polar coordinates, we get

$$\int\limits_{\Omega} |x|^{-\alpha}\, dx = \lim_{\varepsilon \to 0} \int\limits_{\varepsilon \leqslant |x| \leqslant 1} |x|^{-\alpha}\, dx$$

$$= \lim_{\varepsilon \to 0} \int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} \int\limits_{\varepsilon}^{1} r^2 \cos\theta \cdot r^{-\alpha}\, dr\, d\theta\, d\varphi$$

$$= \lim_{\varepsilon \to 0} 2\pi \int\limits_{\varepsilon}^{1} r^{2-\alpha}\, dr \int\limits_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \lim_{\varepsilon \to 0} \frac{4\pi}{3-\alpha}\, r^{3-\alpha}\Big|_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0} \frac{4\pi}{3-\alpha}\,(1 - \varepsilon^{3-\alpha}) = \begin{cases} 4\pi/(3-\alpha), & \alpha < 3, \\ \infty, & \alpha > 3. \end{cases}$$

We have proved that integral (3) is convergent for $\alpha < 3$. If $\alpha > 3$, then integral (3) is divergent. For $\alpha = 3$ inte-

gral (3) also diverges (when computing the integral over r, the antiderivative is equal to $\ln r$).

Remark 1. In the n-dimensional space, i.e. when $|x|^2 = \sum_{j=1}^{n} x_j^2$, integral (3) is convergent for $\alpha < n$ and divergent for $\alpha \geqslant n$.

If the domain G is unbounded and the function $f(x)$ is continuous on the domain $G_R = G \cdot U(O, R)$ for any R (Fig. 61), then the improper integral over an unbounded domain G is defined as the number equal to the limit

$$\lim_{R\to\infty} \int_{G_R} f(x)\,dx = \int_G f(x)\,dx. \tag{4}$$

Example 2. The integral $\int_G |x|^{-\alpha}\,dx$, where $G = E^3 \setminus U(O, 1)$, is convergent for $\alpha > 3$ and divergent for $\alpha \leqslant 3$.

Proceeding as in Example 1, we obtain

$$\int_G |x|^{-\alpha}\,dx = \lim_{R\to\infty} \int_{G_R} |x|^{-\alpha}\,dx = \lim_{R\to\infty} 2\pi \int_1^R r^{-\alpha+2}\,dr \int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta$$

$$= \lim_{R\to\infty} \frac{4\pi}{3-\alpha}\left(R^{3-\alpha} - 1\right) = \begin{cases} \infty, & \alpha < 3, \\ 4\pi/(\alpha-3), & \alpha > 3. \end{cases}$$

For $\alpha = 3$

$$\int_G |x|^{-\alpha}\,dx = \lim_{R\to\infty} 4\pi \int_1^R r^{-1}\,dr = \lim_{R\to\infty} 4\pi \ln R = \infty.$$

Remark 2. In the n-dimensional space the integral

$$\int_G |x|^{-\alpha}\,dx, \quad G = E^n \setminus U(O, 1),$$

converges for $\alpha > n$ and diverges for $\alpha \leqslant n$.

Example 3. Investigate the integral

$$I = \int_0^\infty \int_0^\infty \exp(-x_1^2 - x_2^2)\, dx_1\, dx_2.$$

By definition, we have

$$I = \lim_{R\to\infty} \int_{G_R} \exp(-x_1^2 - x_2^2)\, dx_1\, dx_2,$$

where $G_R = \{|x| \leqslant R,\ x_1 > 0,\ x_2 > 0\}$ is a quarter of

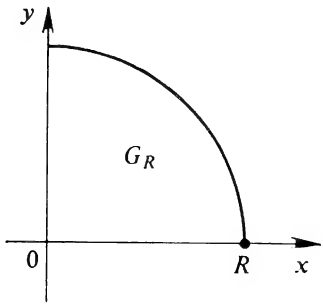

Fig. 62.

a circle of radius R (Fig. 62). Passing to the polar coordinates

$$x_1 = r\cos\varphi,\quad x_2 = r\sin\varphi\quad \left(0 < \varphi < \frac{\pi}{2},\ 0 < r < R\right),$$

$$\frac{D(x_1, x_2)}{D(r, \varphi)} = r,$$

we have

$$\int_{G_R} \exp(-x_1^2 - x_2^2)\, dx_1\, dx_2 = \int_0^R \int_0^{\pi/2} \exp(-r^2)\, r\, dr\, d\varphi$$

$$= \frac{\pi}{2} \int_0^R \exp(-r^2)\, \frac{dr^2}{2} = \frac{-\pi}{4} \exp(-r^2)\Big|_0^R$$

$$= \frac{\pi}{4}\left[1 - \exp(-R^2)\right] \to \frac{\pi}{4},\quad R \to \infty.$$

Thus,

$$\int_0^\infty \int_0^\infty \exp(-x_1^2-x_2^2)\,dx_1\,dx_2=\frac{\pi}{4}.$$

But

$$\int_0^\infty \int_0^\infty e^{-x_1^2-x_2^2}\,dx_1\,dx_2=\left(\int_0^\infty e^{-x_1^2}\,dx_1\right)\left(\int_0^\infty e^{-x_2^2}\,dx_2\right)$$

$$=\left(\int_0^\infty e^{-\xi^2}\,d\xi\right)^2,$$

where the improper integral of one variable on the right (*Poisson's integral*) is convergent. Therefore we obtain

$$\int_0^\infty \exp(-\xi^2)\,d\xi=\frac{\sqrt{\pi}}{2}.$$

Example 4. Compute the integral

$$\int_0^\infty \exp(-\alpha x^2)\,(2\alpha x^2+1)^{-2}\,dx \quad (\alpha>0).$$

Integrating by parts twice, we have

$$\int_\varepsilon^N e^{-\alpha x^2}(2\alpha x^2+1)^{-2}\,dx=\left(u=\frac{e^{-\alpha x^2}}{4\alpha x},\quad dv=\frac{4\alpha x\,dx}{(2\alpha x^2+1)^2}\right)$$

$$=-\frac{e^{-\alpha x^2}}{4\alpha x\,(2\alpha x^2+1)}\Bigg|_\varepsilon^N+\int_\varepsilon^N \frac{1}{2\alpha x^2+1}\,e^{-\alpha x^2}\,\frac{(-2\alpha x^2-1)}{4\alpha x^2}\,dx$$

$$=\frac{-e^{-\alpha x^2}}{4\alpha x\,(2\alpha x^2+1)}\Bigg|_\varepsilon^N-\int_\varepsilon^N e^{-\alpha x^2}\,\frac{dx}{4\alpha x^2}=\left(u=e^{-\alpha x^2},\quad dv=\frac{dx}{x^2}\right)$$

$$=\frac{-e^{-\alpha x^2}}{4\alpha x\,(2\alpha x^2+1)}\Bigg|_\varepsilon^N-\frac{1}{4\alpha}\left[-e^{-\alpha x^2}\,\frac{1}{x}\Bigg|_\varepsilon^N\right.$$

$$\left.-2\alpha\int_\varepsilon^N e^{-\alpha x^2}\,x\,\frac{dx}{x}\right]=\frac{x}{2}\,\frac{e^{-\alpha x^2}}{2\alpha x^2+1}\Bigg|_\varepsilon^N+\frac{1}{2}\int_\varepsilon^N e^{-\alpha x^2}\,dx.$$

Passing to the limit, as $\varepsilon \to 0$ and $N \to \infty$, we shall have

$$\int_0^\infty e^{-\alpha x^2}(2\alpha x^2+1)^{-2}\,dx = \frac{1}{2}\int_0^\infty e^{-\alpha x^2}\,dx = (\sqrt{\alpha}\,x = z)$$

$$= \frac{1}{2}\int_0^\infty e^{-z^2}\frac{dz}{\sqrt{\alpha}} = \frac{1}{2\sqrt{\alpha}}\cdot\frac{\sqrt{\pi}}{2} = \frac{1}{4}\sqrt{\frac{\pi}{\alpha}}.$$

Sec. 2.14. Improper Integral wtih Singularities Along a Line

Let a function $F(x, y)$ be continuous on an open circle

$$G_a = \{x^2 + y^2 < a^2\} \quad (a > 0),$$

but unbounded on it. We assume here that, when approaching any points of the circle $x^2 + y^2 = a^2$, the function F tends to infinity.

Then for any positive $b < a$ the integral

$$\iint_{G_b} F(x, y)\,dx\,dy = \iint_{\bar{G}_b} F(x, y)\,dx\,dy$$

is existent, but the integral of F over G_a in the usual (Riemann) sense does not exist. We know that the existence of integral over G_a in the Riemann sense implies the boundedness of F on G_a.

But it may happen that there exists the limit

$$\lim_{\substack{b\to a\\ b<a}} \iint_{G_b} F(x, y)\,dx\,dy = I.$$

The limit I is called an integral of F over G_a in the improper sense and is denoted as an ordinary Riemann integral:

$$I = \iint_{G_a} F(x, y)\,dx\,dy.$$

We encountered such a situation when considering Example 1 in Sec. 2.11. The integrand of the integral in question is continuous on an open circle G_a, but is unbounded on it.

We had to determine the area of the sphere $|S_a|$ corresponding to G_a with the aid of not an ordinary Riemann integral, but the improper Riemann integral

$$|S_a| = \lim_{b \to a} \iint\limits_{G_b} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \iint\limits_{G_a} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy.$$

We have considered an example of the improper integral when the integrand is unbounded along a line. In the preceding section we gave several examples of improper integrals where the integrand was unbounded in a neighbourhood of a point.

Sec. 2.15. Improper Integral Depending on a Parameter *

Let us consider the improper integral

$$F(x) = \int\limits_{\Omega} f(x, y)\, dy = \int \ldots \int\limits_{\Omega} f(x_1, \ldots, x_m; y_1, \ldots, y_n)\, dy_1 \ldots dy_n, \quad (1)$$

depending on the parameter $x = (x_1, \ldots, x_m)$. We shall assume that the integral has only one singularity at the point $y^0 = (y_1^0, \ldots, y_n^0) \in \bar{\Omega}$.

To be more precise, we consider the domain Ω of points $y = (y_1, \ldots, y_n)$ of the n-dimensional space over which integration takes place, the domain G of points $x = (x_1, \ldots, \ldots, x_m)$ being a domain of parameters. Since we integrate over Ω, and later on we shall also integrate over G, we assume that both domains Ω and G are bounded and have a piecewise smooth boundary. As to the function $f(x, y)$, it is supposed that it is continuous on $\bar{G} \times \bar{\Omega}$** except for the points (x, y^0) where it has a singularity.

* In Sec. 2.15 we shall denote the points x, y in space in lightface, but not by letters in bold type.

** $G \times \Omega$ denotes the direct product of the sets G and Ω, i.e. this is the set of all the pairs (x, y), where $x \in G$, $y \in \Omega$.

The function $f(x, y)$ is, generally speaking, unbounded on Ω in a neighbourhood of every point (x, y^0).

We assume that improper integral (1) exists for all $x \in \overline{G}$. This means that for every $x \in \overline{G}$ there exists a finite limit

$$F(x) = \lim_{\varepsilon \to 0} F_\varepsilon(x) = \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} f(x, \quad y)\, dy = \int\limits_{\Omega} f(x, y)\, dy, \qquad (2)$$

where

$$F_\varepsilon(x) = \int\limits_{\Omega_\varepsilon} f(x, y)\, dy \qquad (3)$$

and $\Omega_\varepsilon = \Omega \setminus U(y^0, \varepsilon)$ is a set of points $y \in \Omega$ except for a ball of radius ε with centre at the point y^0.

It is important to note that integral (3) is an ordinary (proper) Riemann integral, and since the function $f(x, y)$ is continuous on $\overline{G} \times \Omega_\varepsilon$ for any $\varepsilon > 0$, the following known properties are fulfilled for this integral:

(1) $F_\varepsilon(x)$ is a continuous function on $x \in \overline{G}$.

(2) It is legitimate to change the order of integration

$$\int\limits_G dx \int\limits_{\Omega_\varepsilon} f(x, \quad y)\, dy = \int\limits_{\Omega_\varepsilon} dy \int\limits_G f(x, y)\, dx, \qquad (4)$$

(3) It is legitimate to differentiate under the integral sign

$$\frac{\partial}{\partial x_j} \int\limits_{\Omega_\varepsilon} f(x, \quad y)\, dy = \int\limits_{\Omega_\varepsilon} \frac{\partial}{\partial x_j} f(x, y)\, dy, \qquad (5)$$

under the additional condition that the partial derivative $\frac{\partial}{\partial x_j} f(x, y)$ is continuous on $\overline{G} \times \Omega_\varepsilon$.

There arises a question whether properties (1)-(3) are valid for $\varepsilon = 0$, i.e. whether they hold for improper integral (1). Generally speaking, it is not so. But if the convergence of $F_\varepsilon(x)$ to $F(x)$ and $\frac{\partial}{\partial x_j} F_\varepsilon$ to $\frac{\partial}{\partial x_j} F$ is superimposed with the additional condition of uniform convergence, then properties (1)-(3) are retained. In this connection the notion of a uniform convergence of an improper integral is useful.

By definition, *integral* (1) *converges uniformly on* $\overline{G}$ (*or over* $x \in \overline{G}$), *if*

$$F_\varepsilon(x) \to F(x) \quad (\varepsilon \to 0),$$

i.e.

$$\int\limits_{\Omega_\varepsilon} f(x, y)\, dy \to \int\limits_{\Omega} f(x, y)\, dy \quad (\varepsilon \to 0)$$

uniformly on $\overline{G}$.

In other words, integral (1) *is uniformly convergent on* $\overline{G}$ *if the following condition is fulfilled: for any* $\eta > 0$ *there is* $\varepsilon_0 > 0$ *such that*

$$|F(x) - F_\varepsilon(x)| = \left| \int\limits_{\Omega} f(x, y)\, dy - \int\limits_{\Omega_\varepsilon} f(x, y)\, dy \right|$$

$$= \left| \int\limits_{\Omega \cap U(y^0,\ \varepsilon)} f(x, y)\, dy \right| < \eta,$$

$$\forall \varepsilon < \varepsilon_0, \quad \forall x \in \overline{G}.$$

We may apply the theory of uniformly convergent sequences of functions connected with the theory of uniformly convergent series to uniformly convergent integrals as well.

We know that if a sequence of functions $F_n(x)$ $(n = 1, 2, \ldots)$ continuous on a set $\overline{G}$ converges uniformly on $\overline{G}$, then the limit function $F(x)$ is continuous on $\overline{G}$ and

$$\lim_{n \to \infty} \int\limits_{G} F_n(x)\, dx = \int\limits_{G} F(x)\, dx. \tag{6}$$

We also know that if we suppose in addition that the partial derivatives $\frac{\partial}{\partial x_j} F_n(x)$ are existent and continuous on $\overline{G}$ and, besides,

$$\frac{\partial}{\partial x_j} F_n(x) \to \psi(x), \quad x \in \overline{G},$$

uniformly on $\overline{G}$, then the function $F(x)$ has the derivative $\frac{\partial}{\partial x_j} F(x)$, equal to $\psi(x)$:

$$\frac{\partial}{\partial x_j} F(x) = \psi(x), \quad \forall x \in \overline{G}.$$

In proving these properties, the fact that n, increasing, runs through natural numbers is of no importance. We may also assume that $n = \varepsilon$ tends continuously to zero ($\varepsilon \to 0$). Therefore the indicated properties are automatically applied to uniformly convergent improper integrals.

Theorem 1. *If integral* (1) *is uniformly convergent on* $\overline{G}$ *and the function* $f(x, y)$ *is continuous on* $\overline{G} \times \overline{\Omega}$, *except for the points* (x, y^0), *then integral* (1) *is a continuous function of* x. *And*

$$\int_G dx \int_\Omega f(x, y)\, dy = \int_\Omega dy \int_G f(x, y)\, dx.$$

Indeed, from the continuity of $F_\varepsilon(x)$, $\forall \varepsilon > 0$, and uniform convergence of $F_\varepsilon(x) \to F(x)$, $\varepsilon \to 0$, on $\overline{G}$, it follows that $F(x)$ is continuous on $\overline{G}$. Further

$$\int_G dx \int_\Omega f(x, y)\, dy = \int_G F(x)\, dx = \lim_{\varepsilon \to 0} \int_G F_\varepsilon(x)\, dx$$

$$= \lim_{\varepsilon \to 0} \int_G dx \int_{\Omega_\varepsilon} f(x, y)\, dy = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} dy \int_G f(x, y)\, dx$$

$$= \int_\Omega dy \int_G f(x, y)\, dx.$$

In the second equality of this chain we have taken advantage of the formula

$$\int_G F_\varepsilon(x)\, dx \to \int_G F(x)\, dx \quad (\varepsilon \to 0),$$

which is true since F_ε and F are continuous on G and $F_\varepsilon \to F$ uniformly on $\overline{G}$, and formula (4) (in the fourth equality).

Theorem 2. *If, in addition to the fulfilment of the conditions of Theorem* 1, *it is known that the partial derivative* $\frac{\partial}{\partial x_j} f(x, y)$ *is continuous on* $\overline{G} \times \overline{\Omega}$, *except for the points* (x, y^0), *and the integral*

$$\int_\Omega \frac{\partial}{\partial x_j} f(x, y)\, dy$$

converges uniformly on $\bar{G}$, *then the following equality takes place*

$$\frac{\partial}{\partial x_j}\int\limits_{\Omega} f(x, y)\,dy = \int\limits_{\Omega} \frac{\partial}{\partial x_j} f(x, y)\,dy, \tag{7}$$

i.e. it is legitimate to differentiate under the integral sign.

Indeed,

$$\frac{\partial}{\partial x_j}\int\limits_{\Omega} f(x, y)\,dy = \frac{\partial}{\partial x_j} F(x) = \lim_{\varepsilon\to 0}\frac{\partial}{\partial x_j} F_\varepsilon(x)$$

$$= \lim_{\varepsilon\to 0}\frac{\partial}{\partial x_j}\int\limits_{\Omega_\varepsilon} f(x, y)\,dy = \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon}\frac{\partial}{\partial x_j} f(x, y)\,dy$$

$$= \int\limits_{\Omega}\frac{\partial}{\partial x_j} f(x, y)\,dy.$$

In the second equality of this chain we have used the following property: if the functions $F(x)$ and $\frac{\partial}{\partial x_j}F_\varepsilon(x)$ are continuous on $\bar{G}$ and both of them converge uniformly on $\bar{G}$ to $F(x)$ and $\psi(x)$, respectively, then $\frac{\partial}{\partial x_j}F(x) = \psi(x)$ on $\bar{G}$. In the fourth equality we have applied property (5) which is true for any $\varepsilon > 0$.

The following theorem gives a sufficient test for uniform convergence of improper integral (1).

Theorem 3 (Weierstrass' test). *If a function* $f(x, y)$ *is continuous on* $\bar{G}\times\bar{\Omega}$, *except for the points* (x, y^0) $(y^0\in\bar{\Omega})$, *and satisfies on* $\bar{G}\times\bar{\Omega}$ *the inequality*

$$|f(x, y)|\leqslant\varphi(y)\quad (x\in\bar{G}), \tag{8}$$

where the integral

$$\int\limits_{\Omega}\varphi(y)\,dy < \infty \tag{8'}$$

is convergent, then integral (1) *converges uniformly on* $\bar{G}$.

Proof. Let us take $\eta > 0$. By virtue of the convergence of integral (8'), there can be found $\varepsilon_0 > 0$ such that

$$\left|\int\limits_{\Omega\cap U(y^0,\,\varepsilon)}\varphi(y)\,dy\right| = \left|\int\limits_{\Omega\setminus\Omega_\varepsilon}\varphi(y)\,dy\right| < \eta,\quad \forall\varepsilon < \varepsilon_0,$$

therefore if $\varepsilon < \varepsilon' < \varepsilon_0$, then

$$\eta > \left| \int\limits_{\Omega \setminus \Omega_{\varepsilon'}} \varphi(y)\, dy - \int\limits_{\Omega \setminus \Omega_{\varepsilon}} \varphi(y)\, dy \right| = \left| \int\limits_{\Omega_{\varepsilon} \setminus \Omega_{\varepsilon'}} \varphi(y)\, dy \right|,$$

$$\forall \varepsilon,\ \varepsilon' < \varepsilon_0.$$

But then, by virtue of (8),

$$\eta > \left| \int\limits_{\Omega_{\varepsilon} \setminus \Omega_{\varepsilon'}} \varphi(y)\, dy \right| > \left| \int\limits_{\Omega_{\varepsilon} \setminus \Omega_{\varepsilon'}} f(x, y)\, dy \right| = | F_{\varepsilon}(x) - F_{\varepsilon'}(x) |$$

for any $\varepsilon,\ \varepsilon' < \varepsilon_0$ and any $x \in \overline{G}$.

We have obtained Cauchy's condition for the uniform convergence of $F_{\varepsilon}(x)$ on $\overline{G}$, as $\varepsilon \to 0$. This shows that uniform convergence takes place

$$\lim_{\varepsilon \to 0} F_{\varepsilon}(x) = \lim_{\varepsilon \to 0} \int\limits_{\Omega_{\varepsilon}} f(x, y)\, dy = \int\limits_{\Omega} f(x, y)\, dy.$$

Example 1. The integral

$$\psi(a) = \int\limits_0^1 x^{a-1}\, dx \quad (a > 0) \tag{9}$$

exists for any $a > 0$. For $0 < a < 1$ the point $x = 0$ is singular, and for $a \geqslant 1$ the integrand is continuous on the closed interval $[0, 1]$ and the integral has no singularities.

To find out whether the improper integral is uniformly convergent, we must estimate the integral

$$\left| \int\limits_0^{\varepsilon} x^{a-1}\, dx \right| = \frac{\varepsilon^a}{a},$$

which is used to be called the remainder of the integral corresponding to the singular point $x = 0$. For an arbitrary $\eta > 0$ it is impossible to choose $\varepsilon_0 > 0$ so that the remainder is less than η, $\forall a > 0$, $\forall \varepsilon < \varepsilon_0$, since for any fixed ε

$$\lim_{a \to 0} \varepsilon^a / a = \infty \quad (a > 0).$$

Therefore, integral (9) converges non-uniformly with respect to $a > 0$.

Hence, the function $x^{a-1} = \varphi(x, a)$ is continuous on $[0, 1] \times [0, 1]$ except for the points with $x = 0$. The integral $\psi(a) = 1/a$ itself is a discontinuous function on $[0, 1]$. This shows that the requirement of the uniform convergence of the integral is essential for its continuity.

Further it is obvious that for $0 < a_0 \leqslant a \leqslant 1$ integral (9) converges uniformly. Indeed, if $a_0 \leqslant a$, then on the interval $[0, \varepsilon]$, where $0 < \varepsilon < 1$, $x^{a-1} \leqslant x^{a_0-1}$, and the integral

$$\int_0^{\varepsilon} x^{a_0-1}\,dx = \frac{\varepsilon^{a_0}}{a_0} < \infty;$$

therefore, by Weierstrass' test, integral (9) converges uniformly for $a_0 \leqslant a \leqslant 1$.

Thus, integral (9) is a continuous function on $[a_0, 1]$.

If $a > 0$, then integral (9) may be differentiated under the integral sign, i.e.

$$\psi'(a) = \int_0^1 \frac{\partial}{\partial a} x^{a-1}\,dx = \int_0^1 x^{a-1} \ln x\,dx. \tag{10}$$

Indeed, since $\lim\limits_{x \to 0} x^{\lambda} \ln x = 0$ $(\lambda > 0)$, the function

$$\mu(x) = \begin{cases} x^{\lambda} \ln x, & x > 0, \\ 0, & x = 0 \end{cases}$$

is continuous on $[0, 1]$ and, hence, is bounded on $[0, 1]$. Therefore for $0 < a_0 \leqslant a \leqslant 1$

$$| x^{a-1} \ln x | \leqslant | x^{\alpha-\lambda-1} | \cdot | x^{\lambda} \ln x | \leqslant c\, |x^{\alpha-\lambda-1}|$$

and the integral

$$\int_0^{\varepsilon} x^{a-\lambda-1}\,dx = \frac{\varepsilon^{a-\lambda}}{a-\lambda} < \infty \quad (0 < \lambda \leqslant a - a_0).$$

Hence, by Weierstrass' test, the integral on the right in (10) is uniformly convergent. Further, the function $\frac{\partial}{\partial a} x^{a-1} = x^{a-1} \ln x$ is continuous on $[0, 1] \times [0, 1]$, except for the points with $x = 0$, therefore, according to Theorem 2, formula (10) is correct.

Integral (1) can be considered for an unbounded domain Ω, assuming that the function $f(x, y)$ is continuous on the set $\overline{G} \times \Omega$ of points (x, y). Since the domain Ω is unbounded, integral (1) does not exist as a Riemann integral, but it can exist in the improper sense as the limit

$$\lim_{R\to\infty} F_R(x) = \lim_{R\to\infty} \int\limits_{\Omega_R} f(x, y)\, dy = \int\limits_{\Omega} f(x, y)\, dy = F(x), \qquad (11)$$

where $\Omega_R = \Omega \cap U(O, R)$ $U(O, R)$ is a ball of radius R with centre at the zero point.

We hold that[1]

$$F_R(x) = \int\limits_{\Omega_R} f(x, y)\, dy$$

and that there exists the limit

$$\lim_{R\to\infty} F_R(x) = F(x) = \int\limits_{\Omega} f(x, y)\, dy$$

for all $x \in \overline{G}$, i.e. (11), as an improper integral exists for all $x \in \overline{G}$.

In this case we say that integral (11) has only one singularity at the point at infinity.

Integral (11) is defined as uniformly convergent if $F_R(x) \to F(x)$, $R \to \infty$, uniformly with respect to $x \in \overline{G}$.

The same as in the case of a finite singular point, it is proved that for a function $f(x, y)$ continuous on $\overline{G} \times \Omega$, integral (11) is a continuous function of $x \in \overline{G}$ if it is uniformly convergent on $\overline{G}$. Further, if G is a bounded domain with a piecewise smooth boundary, then the following equality takes place:

$$\int\limits_{G} dx \int\limits_{\Omega} f(x, y)\, dy = \int\limits_{\Omega} dy \int\limits_{G} f(x, y)\, dx.$$

If $\frac{\partial f}{\partial x_j}$ is continuous on Ω and the integral of $\frac{\partial f}{\partial x_j}$ over Ω converges uniformly with respect to $x \in \overline{G}$, then (11) can be differentiated under the integral sign

$$\frac{\partial}{\partial x_j} \int\limits_{\Omega} f(x, y)\, dy = \int\limits_{\Omega} \frac{\partial}{\partial x_j} f(x, y)\, dy.$$

Example 2. Investigate the integral

$$I(x) = \int_0^\infty xe^{-xy}\,dy \quad (x \geqslant 0).$$

It is obvious that $I(0) = 0$, and for $x \neq 0$

$$I(x) = \lim_{R\to\infty} \int_0^R xe^{-xy}\,dy = \lim_{R\to\infty} -e^{-xy}\Big|_0^R = 1.$$

Thus, the function $I(x)$ is discontinuous on $[0, \infty)$. This is connected with the fact that our integral converges non-uniformly: the remainder of the integral

$$\int_R^\infty xe^{-xy}\,dy = e^{-Rx}$$

does not tend to zero for any fixed R and $x \to 0$ (it tends to 1).

Example 3. Gamma function. The integral

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\,dx \tag{12}$$

is called the *gamma function* or *Euler's integral of the second kind.*

For $a \geqslant 1$ it has the point $x = \infty$ as its singularity and for $0 < a < 1$ it has two singular points: $x = 0$ and $x = \infty$. Therefore in order to investigate this integral, it is advisable to split it into two integrals

$$\Gamma(a) = \int_0^1 x^{a-1}e^{-x}\,dx + \int_1^\infty x^{a-1}e^{-x}\,dx.$$

Since for $0 < x \leqslant 1$ we have $x^{a-1}e^{-x} \leqslant x^{a-1}$, then (as is seen from Example 1) the first integral is uniformly convergent for all $a \geqslant a_0 > 0$, where a_0 is an arbitrary fixed positive number. It is obvious, that the second integral is convergent for any real a.

If a_0 is an arbitrary number, then for $a \leqslant a_0$ we have

$$x^{a-1}e^{-x} \leqslant x^{a_0-1}e^{-x} \quad (1 \leqslant x < \infty),$$

and since

$$\int_1^\infty x^{a_0-1}e^{-x}\,dx < \infty,$$

we see that, by Weierstrass' test, the second integral is uniformly convergent for all $a \leqslant a_0$. At the same time, it is not uniformly convergent on the whole real axis $-\infty < a < \infty$.

For instance, for $a > 1$ and $R > 1$

$$\int_R^\infty x^{a-1}e^{-x}\,dx \geqslant R^{a-1}\int_R^\infty e^{-x}\,dx = R^{a-1}e^{-R} \to \infty$$

for $a \to +\infty$ and any fixed R.

For $a > 1$

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\,dx$$

$$= \lim_{R\to\infty}\left\{-x^{a-1}e^{-x}\Big|_0^R + (a-1)\int_0^R x^{a-2}e^{-x}\,dx\right\}$$

$$= (a-1)\,\Gamma(a-1).$$

Therefore for natural values $a = n$.

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\,\Gamma(n-1) = \ldots = n!\,\Gamma(1)$$

$$= n!\int_0^\infty e^{-x}\,dx = n!,$$

whence it is seen that the gamma function is a natural generalization of the factorial.

CHAPTER 3

VECTOR ANALYSIS

Sec. 3.1. A Piecewise Smooth Oriented Curve

The curve

$$\mathbf{r}(t) = \varphi(t)\,\mathbf{i} + \psi(t)\,\mathbf{j} + \chi(t)\,\mathbf{k} \quad (a \leqslant t \leqslant b) \qquad (1)$$

is said to be *piecewise smooth continuous* if the functions φ, ψ, χ, are continuous on $[a, b]$ and the interval $[a, b]$ can be separated into a finite number of subintervals by the division points

$$a = t_0 < t_1 < \ldots < t_N = b$$

so that on each of them the functions φ, ψ, χ have continuous derivatives which are not all zero*.

Figure 63 represents a piecewise smooth continuous curve. At the points A_1 and A_2 it is continuous, but the derivatives $\varphi'(t)$, $\psi'(t)$, $\chi'(t)$ (all or some of them) have discontinuities (of the first kind!).

Curve (1) will be denoted by one letter, say Γ. The letter Γ usually denotes not only the locus of points (x, y, z), defined by equations (1), but also the order in which these points follow when t increases continuously from a to b $(a < b!)$. In this sense Γ is said to be an *oriented curve*. This order is indicated by an arrow (see Fig. 63): when t increases continuosly from a to b, the point (x, y, z) moves along Γ in the direction indicated by the arrow.

If $t = \lambda(t)$ is a function having a continuous positive derivative on a certain interval $[c, d]$ and $\lambda(c) = a$, $\lambda(d) = b$, then the equation

$$\mathbf{r}(\lambda(\tau)) = \varphi[\lambda(\tau)]\,\mathbf{i} + \psi[\lambda(\tau)]\,\mathbf{j} + \chi[\lambda(\tau)]\,\mathbf{k} \quad (c \leqslant \tau \leqslant d) \qquad (1')$$

---

* See our book *Higher Mathematics. Differential and Integral Calculus*, Secs. 4.21, 7.3.

defines the same oriented curve as Γ. It is denoted by the same letter Γ but in the case of equation (1) we say that Γ is defined by the parameter t, while in the case of equation (1′) by the parameter τ. In both cases when t increases from a to b or τ increases from c to d the corresponding points of Γ move in one and the same direction.

Quite another matter if we make the substitution $t = \lambda(\tau)$, where $\lambda(\tau)$ has a continuous negative derivative

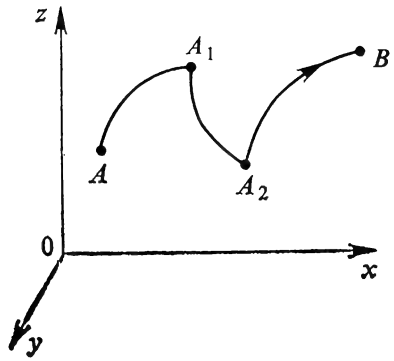

Fig. 63.

on the interval $[c, d]$ ($c < d$!). In this case $\lambda(c) = b$, $\lambda(d) = a$, and as t increases continuously from c to d, the parameter t will decrease and the arrow should be reversed on our geometrical object.

Therefore, if $\lambda'(\tau) < 0$, we shall denote the curve (1′) by another symbol Γ_- and say that Γ_- is the same curve as Γ but oriented in the opposite direction. The originally oriented curve will sometimes be denoted by the symbol Γ_+.

Oriented curve (1) is called *closed* or a *closed contour* if $\mathbf{r}(a) = \mathbf{r}(b)$ or, which is the same, if

$$\varphi(a) = \varphi(b), \quad \psi(a) = \psi(b), \quad \chi(a) = \chi(b).$$

In other words, when the value of the parameter t increases continuously from a to b the corresponding point (x, y, z) describes in space a closed path which begins and ends at one and the same point. And if the curve Γ does not intersect itself at other points, then it is called a closed non-self-

intersecting curve (see Fig. 64). A closed self-intersecting curve is shown in Fig. 65.

Remark. Vector equation (1) of the curve Γ is equivalent to the three equations

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t) \quad (a \leqslant t \leqslant b).$$

In this case we shall say that the curve Γ is represented parametrically.

Fig. 64.

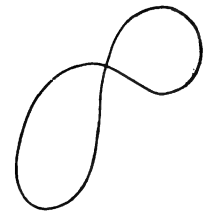

Fig. 65.

Example 1. Let the curve Γ be given by the equation

$$\mathbf{r}(t) = \mathbf{i}R\cos t + \mathbf{j}R\sin t \quad (x = R\cos t,\ y = R\sin t),$$
$$0 \leqslant t \leqslant 2\pi.$$

Since $x^2 + y^2 = R^2(\cos^2 t + \sin^2 t) = R^2$, the given curve is a circle of radius R centred at the origin. As t increases from 0 to 2π, the point $A = (x, y)$ moves in the circle anticlockwise. And to different values of t there correspond distinct points A. For $t = 0$ and $t = 2\pi$ we have $\mathbf{r}(0) = \mathbf{r}(2\pi) = \mathbf{i}R$. Hence, the circle is a closed non-self-intersecting curve (Fig. 66).

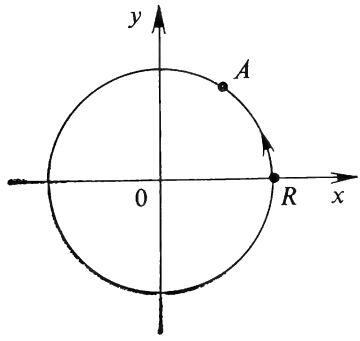

Fig. 66.

Example 2. The curve

$$\mathbf{r}(t) = a(\mathbf{i}\cos t + \mathbf{j}\sin t) + bt\,\mathbf{k},$$

where $0 \leqslant t < \infty$, a, b are positive numbers is called a helix. It can be obtained in the following way: a line segment of length a perpendicular to the z-axis slides with one of its

ends along the z-axis and, simultaneously, turns about this axis, then the other end describes a helix. We assume that the height to which the segment climbs along the z-axis

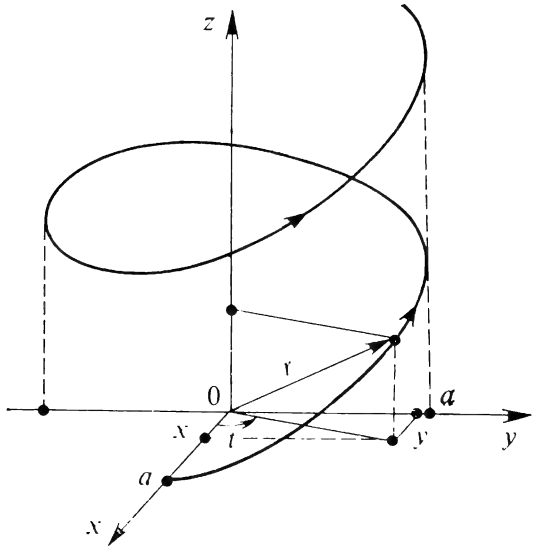

Fig. 67.

is proportional to the angle of rotation t ($z = bt$). With an increase in t the point (x, y, z) moves as is shown in Fig. 67. It is obvious that the helix lies on the lateral surface of a circular cylinder of radius a with the generatrix parallel to the z-axis.

Sec. 3.2. Line Integral of the First Type

Let there be given a continuous piecewise smooth curve Γ

$$\mathbf{r}(t) = \varphi(t)\,\mathbf{i} + \psi(t)\,\mathbf{j} + \chi(t)\,\mathbf{k} \quad (0 \leqslant t \leqslant T), \qquad (1)$$

and let a continuous function $F(x, y, z)$ be defined on Γ or in the neighbourhood of Γ.

Then the *line integral of the first type of the function* $F(x, y, z)$ *along the curve* Γ is defined as a number equal to

$$\int_{\Gamma} F(x, y, z)\,ds$$

$$= \int_0^T F[\varphi(t), \psi(t), \chi(t)] \sqrt{\varphi'(t)^2 + \psi'(t)^2 + \chi'(t)^2}\,dt. \qquad (2)$$

The left-hand side of (2) is the designation of the integral of the first type, the right-hand side being its definition; this is an ordinary definite integral with respect to t on $[0, T]$.

For instance, if the curve Γ has a mass with a density of distribution $F(x, y, z)$ at points $(x, y, z) \in \Gamma$, then the total mass M of the curve is computed by means of integral (2). But the element of a material curve, corresponding to the interval $[t, t + dt]$ of variation of t, has a mass equal to (with an accuracy to an infinitesimal of a higher order)

$$F\,ds = F[\varphi(t), \psi(t), \chi(t)]\sqrt{\varphi'(t)^2 + \psi'(t)^2 + \chi'(t)^2}\,dt,$$

where ds is the differential of the arc of Γ. This shows that M is equal to the right-hand member of (2).

The value of the integral of the first type remains unchanged when reversing the orientation of the curve

$$\int_{\Gamma} F(x, y, z)\,ds = \int_{\Gamma_-} F(x, y, z)\,ds.$$

For instance, curve (1) can be specified by the equations

$$x = \varphi(T - \tau),\ y = \psi(T - \tau),\ z = \chi(T - \tau) \qquad (0 \leqslant \tau \leqslant T),$$

which orient it in the opposite sense and then

$$\int_0^T F[\varphi(T-\tau), \psi(T-\tau), \chi(T-\tau)] \times \sqrt{\varphi'(T-\tau)^2 + \psi'(T-\tau)^2 + \chi'(T-\tau)^2}\,d\tau$$

$$= -\int_T^0 F[\varphi(t), \psi(t), \chi(t)]\sqrt{\varphi'(t)^2 + \psi'(t)^2 + \chi'(t)^2}\,dt$$

$$= \int_0^T F[\varphi(t), \psi(t), \chi(t)]\sqrt{\varphi'(t)^2 + \psi'(t)^2 + \chi'(t)^2}\,dt.$$

Example 1. Let along the helix defined in Example 2 given in the preceding section there be distributed masses with density $F(x, y, z) = z^2$. Find the mass M of the portion of the helix when the parameter t varies from 0 to 3 $(0 \leqslant t \leqslant 3)$.

Let us write the equation of our helix in the form

$$x = a\cos t,\quad y = a\sin t,\quad z = bt,\quad (0 \leqslant t \leqslant 3).$$

Then

$$M = \int_0^3 F[x(t), y(t), z(t)] \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

$$= \int_0^3 b^2 t^2 \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2}\, dt$$

$$= b^2 \sqrt{a^2 + b^2} \int_0^3 t^2\, dt = 9b^2 \sqrt{a^2 + b^2}.$$

Example 2. Compute the integral of the first type of the function $F(x, y) = \sqrt{\left(\frac{bx}{a}\right)^2 + \left(\frac{ay}{b}\right)^2}$ over the ellipse

$$x = a\cos t,\quad y = b\sin t \quad (0 \leqslant t \leqslant 2\pi).$$

We have

$$\int_\Gamma F(x, y)\, ds = \int_0^{2\pi} \sqrt{b^2 \cos^2 t + a^2 \sin^2 t}\, \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt$$

$$= \int_0^{2\pi} (b^2 \cos^2 t + a^2 \sin^2 t)\, dt = \pi (a^2 + b^2).$$

Sec. 3.3. Integral of a Vector Along a Curve

Let in the space R_3 with a rectangular coordinate system x, y, z there be given an oriented continuous piecewise smooth curve Γ with the initial point A_0 and end point A_1. If Γ is closed, then A_0 coincides with A_1. Let

$$x = \varphi(t),\quad y = \psi(t),\quad z = \chi(t),\quad (0 \leqslant t \leqslant T)$$

be the equations of Γ and let to the value $t = 0$ there correspond the point A_0 and to $t = T$ the point A_1.

At every interior (not a corner) point A of any smooth piece of Γ there defined identically the unit vector τ of the tangent to Γ in the direction of increasing t's.

Let on Γ or on the set Ω containing Γ there be given a field of a continuous vector (there be given a vector)

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

where P, Q, R are thus continuous functions on Γ (or Ω).

This picture can be imagined best of all in the following way (see Fig. 68): from any point $(x, y, z) \in \Gamma$ (or Ω) a vector $\mathbf{a}$ is drawn whose direction and length depend on this point ($\mathbf{a} = \mathbf{a}(x, y, z)$).

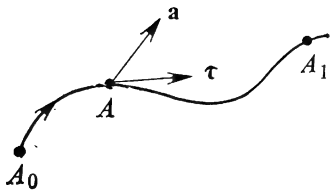

Fig. 68.

The line integral of a vector $\mathbf{a}$ *along an oriented curve* Γ is defined as the quantity

$$\int_{\Gamma} (\mathbf{a}\,d\mathbf{s}) = \int_{\Gamma} (\mathbf{a}\boldsymbol{\tau})\,ds = \int_0^T \{P[\varphi(t), \psi(t), \chi(t)]\,\varphi'(t)$$

$$+ Q[\varphi(t), \psi(t), \chi(t)]\,\psi'(t) + R[\varphi(t), \psi(t), \chi(t)]\,\chi'(t)\}\,dt. \quad (1)$$

The first and second terms in (1) are designations of the new notion, i.e. the integral of $\mathbf{a}$ along (or about) Γ. The third term is its definition —this is an ordinary definite integral with respect to t over $[0, T]$.

The symbol $d\mathbf{s}$ in the first term of (1) should be understood as the vector having the direction of τ and the length equal to the arc length ds of the curve Γ. The symbol $(\mathbf{a}\,d\mathbf{s})$ is the scalar product of $\mathbf{a}$ and $d\mathbf{s}$.

The triple $(\varphi'(t), \psi'(t), \chi'(t))$ is a tangent vector to Γ. Normalizing it, we obtain the unit tangent vector

$$\boldsymbol{\tau} = \left(\frac{\varphi'}{s_t'}, \frac{\psi'}{s_t'}, \frac{\chi'}{s_t'}\right), \quad s_t' = \frac{ds}{dt} = \sqrt{\varphi'(t)^2 + \psi'(t)^2 + \chi'(t)^2}.$$

Therefore

$$(\mathbf{a}\, d\mathbf{s}) = (\mathbf{a}\, \boldsymbol{\tau})\, ds$$
$$= [P(\varphi, \psi, \chi)\, \varphi' + Q(\varphi, \psi, \chi)\psi' + R(\varphi, \psi, \chi)\chi']\, dt.$$

Integrating this element with respect to t, we obtain the right-hand member of (1). The following notation for an integral of vector $\mathbf{a}$ along Γ is also used:

$$\int_{\Gamma} (\mathbf{a}\, d\mathbf{s}) = \int_{\Gamma} [P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz]. \quad (2)$$

It prompts that if it is required to compute the integral of $\mathbf{a}$ along Γ, we have to set in the right-hand side of (2): $x = \varphi(t)$, $y = \psi(t)$, $z = \chi(t)$, $dx = \varphi'(t)\, dt$, $dy = \psi'(t)\, dt$, $dz = \chi'(t)\, dt$ and to integrate the result with respect to t between 0 and T.

The expression

$$\int_{\Gamma} (\mathbf{a}\, d\mathbf{s}) \quad (3)$$

($d\mathbf{s}$ is a vector!) it called the *integral of the second type.*

We should like to underline the following: in order to compute the integral of the second type, we introduce the unit vector $\boldsymbol{\tau}$ of the tangent to Γ directed according to the orientation of Γ and write the integration element in (3) in the form

$$(\mathbf{a}\, d\mathbf{s}) = (\mathbf{a}\boldsymbol{\tau})\, ds.$$

Here ds ($ds > 0$) is a scalar, i.e. the differential of the arc of Γ and the quantity $(\mathbf{a}\boldsymbol{\tau})$ is a function defined on Γ.

The computation of integral (3) of the second type is reduced to that of the integral of the first type:

$$\int_{\Gamma} (\mathbf{a}\, d\mathbf{s}) = \int_{\Gamma} (\mathbf{a}\boldsymbol{\tau})\, ds. \quad (4)$$

The left-hand member of (4) is an integral of the second type, while the right-hand member is an integral of the first type. Obviously, the two integrals are equal to each other.

Thus, there is no essential difference between two types of integrals; it is only the matter of notation on which the terminology depends here.

The expression

$$\int_{\Gamma} (P\,dx + Q\,dy + R\,dz)$$

is also called the integral of the second type.

We know that the integral of the first type along Γ is independent of the orientation of Γ. The integral of the second type does depend on the orientation of Γ. Namely, it changes sign when the orientation is altered:

$$\int_{\Gamma_-} (\mathbf{a}\,d\mathbf{s}) = -\int_{\Gamma} (\mathbf{a}\,d\mathbf{s}).$$

Indeed,

$$\int_{\Gamma_-} (\mathbf{a}\,d\mathbf{s}) = \int_{\Gamma_-} (\mathbf{a}\boldsymbol{\tau}_-)\,ds = \int_{\Gamma} (\mathbf{a}\boldsymbol{\tau}_-)\,ds = -\int_{\Gamma} (\mathbf{a}\boldsymbol{\tau})\,ds = -\int_{\Gamma} (\mathbf{a}\,d\mathbf{s}).$$

Here $\boldsymbol{\tau}$ is the unit vector of the tangent to the oriented curve Γ and $\boldsymbol{\tau}_-$ is the unit vector of the tangent to Γ_-.

The second term in this chain is the integral of the first type of the function $(\mathbf{a}\boldsymbol{\tau}_-)$ (defined on Γ). It does not depend

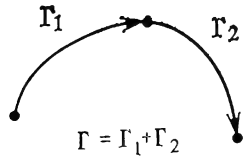

Fig. 69.

on the orientation of Γ which is explained by the second equality of the chain. The third equality follows from the fact that

$$(\mathbf{a}\boldsymbol{\tau}_-) = -(\mathbf{a}\boldsymbol{\tau}).$$

And the last equality of the chain follows from the definition of the integral of the second type of the vector $\mathbf{a}$ along an oriented curve Γ.

If an oriented curve Γ represents the sum of two distinct oriented curves Γ_1 and Γ_2 ($\Gamma = \Gamma_1 + \Gamma_2$), then

$$\int_{\Gamma} (\mathbf{a}\,d\mathbf{s}) = \int_{\Gamma_1} (\mathbf{a}\,d\mathbf{s}) + \int_{\Gamma_2} (\mathbf{a}\,d\mathbf{s}).$$

Figure 69 represents an oriented curve Γ cut into two appropriately oriented curves Γ_1 and Γ_2.

Shown in Fig. 70 are two oriented closed contours Γ_1 and Γ_2. $\Gamma = \Gamma_1 + \Gamma_2$ should be understood as a compound oriented contour, as a union of Γ_1 and Γ_2.

It is assumed by definition that

$$\int_{\Gamma} = \int_{\Gamma_1} + \int_{\Gamma_2}.$$

Example 1. Compute the integral of the second type (Fig. 71)

$$I_1 = \int_{AB} (x^2\,dx + xy\,dy)$$

along the rectilinear segment joining the point (0, 0) to

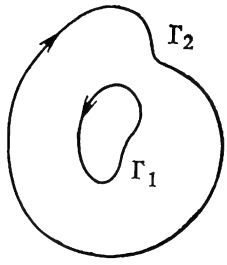

Fig. 70.

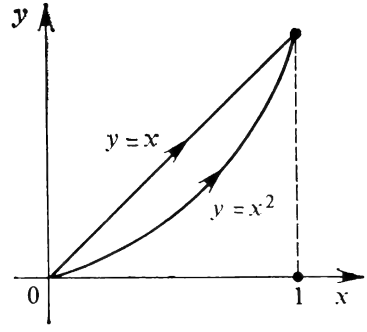

Fig. 71.

the point (1, 1), and along an arc of the parabola $y = x^2$ connecting the same points.

In the first case ($y = x$) we have

$$I_1 = \int_{AB} (x^2\,dx + xy\,dy) = \int_0^1 (x^2 + x^2)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}.$$

In the second case ($y = x^2$)

$$I_1 = \int_0^1 (x^2\,dx + x^3 2x\,dx) = \int_0^1 (x^2 + 2x^4)\,dx = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}.$$

Example 2. Compute the integral of the second type

$$I_2 = \int_{AB} \left(x^2 y\, dx + \frac{x^3}{3}\, dy \right)$$

over the same curves as in Example 1.

We have $(y = x)$

$$I_2 = \int_0^1 \left(x^3\, dx + \frac{x^3}{3}\, dx \right) = \frac{1}{4} + \frac{1}{12} = \frac{1}{3}.$$

Further, for $y = x^2$ we obtain

$$I_2 = \int_0^1 \left(x^4\, dx + \frac{x^3}{3} 2x\, dx \right) = \int_0^1 \left(x^4 + \frac{2}{3} x^4 \right) dx$$

$$= \frac{5}{3} \int_0^1 x^4\, dx = \frac{1}{3}.$$

These examples show that the integral of the second type, generally speaking, depends on the curve along which it is computed, or in other words, on the path of integration.

In the second example we obtained one and the same value along different paths of integration. This turns out to be not accidental. The cause of this property will be explained below.

Sec. 3.4. Potential Field

Of importance is the case of the field of the vector

$$\mathbf{a} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

when on the domain Ω where the field is defined there exists a function $U(x, y, z)$ having continuous partial derivatives for which the equalities

$$\frac{\partial U}{\partial x} = P, \quad \frac{\partial U}{\partial y} = Q, \quad \frac{\partial U}{\partial z} = R$$

are fulfilled on Ω.

Such function is called a *potential function* or the *potential* of the vector **a** on Ω. We also say that the vector **a** is the *gradient of the function* U and write*

$$\operatorname{grad} U = \frac{\partial U}{\partial x}\,\mathbf{i} + \frac{\partial U}{\partial y}\,\mathbf{j} + \frac{\partial U}{\partial z}\,\mathbf{k} = \mathbf{a}.$$

Example 1. The function

$$U(x,\ y,\ z) = -\frac{1}{r},\quad r = \sqrt{x^2 + y^2 + z^2}$$

is defined throughout the entire space excepting the zero point (0, 0, 0). Its gradient is equal to

$$\operatorname{grad} U = \frac{x}{r^3}\,\mathbf{i} + \frac{y}{r^3}\,\mathbf{j} + \frac{z}{r^3}\,\mathbf{k} = \mathbf{a}.$$

In the equality

$$\mathbf{a} = \frac{1}{r^2}\left(\frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} + \frac{z}{r}\,\mathbf{k}\right)$$

standing in the parentheses is a unit vector directed in the same sense with the radius vector of the point (x, y, z). But then

$$|\mathbf{a}| = \frac{1}{r^2}.$$

These facts may be interpreted in the following way. Found at the zero point is a unit electric charge; at point (x, y, z) is also a unit charge of the same sign. The repelling force between these charges is a vector applied at point (x, y, z) and directed as the radius vector of the point (x, y, z); its magnitude is equal to $1/r^2$.

We see that the vector **a** has the potential $U = -1/r$.

Let us pass to the general properties of the field of a vector having a potential.

Theorem 1. *For the field of a vector* **a** *defined in a domain* Ω *in space to have a potential, it is necessary and sufficient that one of the following two conditions be fulfilled:*

(1) *The integral of the vector* **a** *about any closed (piecewise smooth) path* Γ *belonging to* Ω *is equal to zero.*

(2) *The integral about any (piecewise smooth) path* $\Gamma \subset \Omega$, *connecting any two points of* Ω *is independent of the path of integration.*

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* See our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 8.8.

If $U(x, y, z)$ *is a potential function of the vector* $\mathbf{a}$, *then the integral of* $\mathbf{a}$ *about any path* $\Gamma_{AB} \subset \Omega$, *connecting the points* $A = (x_0, y_0, z_0)$ *and* $B = (x, y, z)$ *is equal to*

$$\int_{AB} (\mathbf{a}\, d\mathbf{s}) = U(x, y, z) - U(x_0, y_0, z_0). \tag{1}$$

Proof. First of all let us prove the equivalence of properties (1) and (2).

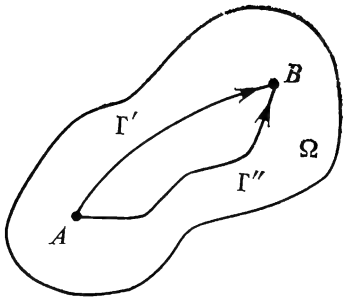

Fig. 72.

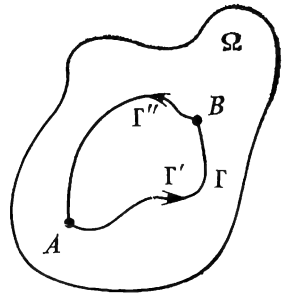

Fig. 73.

Let property (1) be valid. We take in Ω two points A and B (Fig. 72) and connect them with two distinct curves Γ' and Γ'' oriented from A to B.

By virtue of (1),

$$\int_{\Gamma'} + \int_{\Gamma''_-} = 0,$$

therefore

$$\int_{\Gamma'} = -\int_{\Gamma''_-} = \int_{\Gamma''},$$

and we have proved property (2).

Conversely, let property (2) hold. We take an arbitrarily oriented closed contour Γ (Fig. 73) and cut it at points A and B into two properly oriented curves

$$\Gamma = \Gamma' + \Gamma''.$$

By property (2),

$$\int_{\Gamma'} = \int_{\Gamma''_-},$$

whence

$$\int_{\Gamma} = \int_{\Gamma'} + \int_{\Gamma''} = \int_{\Gamma'} - \int_{\Gamma''_-} = 0,$$

and we have proved (1).

Let now be known that the field of the vector **a** has a potential function $U(x, y, z)$ in the domain Ω.

Let us take in Ω a fixed point $A_0 = (x_0, y_0, z_0)$ and a variable point $A = (x, y, z)$ and join A_0 to A with an oriented from A_0 to A continuous piecewise smooth curve $\Gamma = \Gamma_{A_0A}$, defined by the equation

$$x = \varphi(\tau), \quad y = \psi(\tau), \quad z = \chi(\tau) \quad (t_0 \leqslant \tau \leqslant t).$$

Thus, to the values t_0, t of the parameter τ there correspond the points A_0, A.

If we substitute the functions φ, ψ, χ into U instead of x, y, z, respectively, then U will be a continuous piecewise smooth function of τ. By virtue of the theorem on the derivative of a composite function at points where Γ has a tangent (that is, where Γ is smooth),

$$\frac{\partial U}{\partial \tau} = \frac{\partial U}{\partial x}\frac{d\varphi}{d\tau} + \frac{\partial U}{\partial y}\frac{d\psi}{d\tau} + \frac{\partial U}{\partial z}\frac{d\chi}{d\tau}.$$

Hence it follows that

$$\int_{\Gamma} (P\,dx + Q\,dy + R\,dz)$$

$$= \int_{t_0}^{t} \left(\frac{\partial U(\varphi, \psi, \chi)}{\partial x}\varphi'(\tau) + \frac{\partial U(\varphi, \psi, \chi)}{\partial y}\psi'(\tau) \right.$$

$$\left. + \frac{\partial U(\varphi, \psi, \chi)}{\partial z}\chi'(\tau) \right) d\tau = \int_{t_0}^{t} \frac{\partial U}{\partial \tau}\,d\tau = U[\varphi(t), \psi(t), \chi(t)]$$

$$- U[\varphi(t_0), \psi(t_0), \chi(t_0)] = U(x, y, z) - U(x_0, y_0, z_0)$$

$$= U(A) - U(A_0) \equiv V(A), \qquad (2)$$

i.e. the line integral of the second type with the point A_0 fixed depends only on the position of the point $A \in \Omega$, but not on the path along which it is reached from the point A_0. This proves property (2) and equality (1) if it is known that

the vector **a** has a potential function in the domain Ω.

It remains to prove that property (2) implies that there exists a defined on Ω potential function $U(x, y, z)$ whose gradient of Ω is equal to **a**. Indeed, let us take a fixed point $A_0 \in \Omega$ (Fig. 74). Let it be known that property (2) is fulfilled, i.e. the given field of the vector **a** is such that the line integral along any continuous piecewise smooth curve connecting A_0 with an arbitrary point $A \in \Omega$ is independent of this curve and depends only on the point A. Thus, there exists a defined on Ω function $V(A)$ such that

$$\int_{\Gamma_{A_0A}} (P\,dx + Q\,dy + R\,dz) = V(A) = V(x, y, z).$$

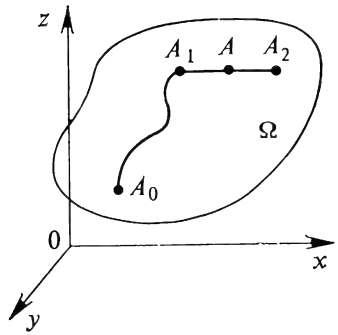

Fig. 74.

To prove that $\frac{\partial V}{\partial x} = P$ at the point A, we shall reason in the following way. We shall join the point A_0 to A with a special curve $\Gamma_{A_0A} \subset \Omega$ (see Fig. 74) which ends in a line segment A_1A parallel to the x-axis. We then extend this segment to some point A_2. Thus, the variable point A of the line segment A_1A_2 has two constant coordinates y and z and only one variable coordinate x. Let us represent the curve Γ_{A_0A} as the sum of two curves

$$\Gamma_{A_0A} = \Gamma_{A_0A_1} + \Gamma_{A_1A},$$

and then

$$V(x, y, z) = \int_{\Gamma_{A_0A}} (\mathbf{a}\,d\mathbf{s}) = \int_{\Gamma_{A_0A_1}} (\mathbf{a}\,d\mathbf{s}) + \int_{\Gamma_{A_1A}} (\mathbf{a}\,d\mathbf{s})$$

$$= K + \int_{x_1}^{x} P(t, y, z)\,dt, \qquad (3)$$

where $K = \int_{\Gamma_{A_0A_1}} (\mathbf{a}\,d\mathbf{s})$ is a constant remaining unchanged when the point A moves in the segment A_1A_2 and $A_1 = (x,$

y, z). It should be taken into consideration that the equations of the segment A_1A_2 can be written in the parametric form (in terms of the parameter t): $x = t$, $y = y$, $z = z$, whence it follows that

$$\int\limits_{\Gamma_{A_1A}} Q\,dy = \int\limits_{x_1}^{x} Q(t, y, z)\cdot 0\,dt = 0,$$

$$\int\limits_{\Gamma_{A_1A}} R\,dz = \int\limits_{x_1}^{x} R(t, y, z)\cdot 0\,dt = 0.$$

Thus, we have obtained equality (3) which is correct whatever the point of the segment A_1A_2 may be. Here y, z are fixed and x is variable. Since the integrand of the right-hand member of (3) is a continuous function of t, then

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\int\limits_{x_0}^{x} P(t, y, z)\,dt = P(x, y, z).$$

Analogously, we can prove that

$$\frac{\partial V}{\partial y} = Q(x, y, z), \quad \frac{\partial V}{\partial z} = R(x, y, z) \quad ((x, y, z) \in \Omega),$$

by introducing special curves $\Gamma_{A_0A} \subset \Omega$, ending in a line segment parallel to the y-axis in one case and to the z-axis in the other.

Example 2. If the force **a** is constant (by magnitude and direction) and the path AB is rectilinear, then, as is known from physics, the work done by the force is defined as the product of the magnitude of this force by the length of the path and by the cosine of the angle between the directions of the force and displacement, i.e. the work done during this displacement is equal to the scalar product $(\mathbf{a}, \overrightarrow{AB})$.

But if the force $\mathbf{a} = \{P, Q, R\}$ is variable and the path is curvilinear, then, obviously, the element of the work done by this force over an elementary interval of the path corresponding to the variation of the parameter from t to $t + \Delta t$ will be approximately equal to the scalar product of the force **a** at some point of this interval and the displacement vector representing a chord connecting the end points

of the elementary interval:

$$P\,\Delta x + Q\,\Delta y + R\,\Delta z.$$

Summing up these elements of work and passing to the limit (as $\Delta t \to 0$), we obtain the work done along the entire curve Γ

$$\int_0^T [P\varphi'(t) + Q\psi'(t) + R\chi'(t)]\,dt = \int_\Gamma (\mathbf{a}\tau)\,ds = \int_\Gamma (\mathbf{a}\,d\mathbf{s}).$$

Thus, the *line integral of the second type represents the work done by a variable force* (*vector* **a**) *along an oriented path.*

Let us compute the work done by the force **a** defined in Example 1 along the path connecting the points (1, 2, 2) and (3, 0, 4).

The force **a** has the potential function

$$U(x, y, z) = -1/r = -1/\sqrt{x^2+y^2+z^2}$$

in the domain Ω which represents the entire space with the zero point deleted. Therefore the line integral is independent of the path.

By virtue of formula (1), the integral of vector **a** along any piecewise smooth path $\Gamma \subset \Omega$ connecting the points (1, 2, 2) and (3, 0, 4) is equal to

$$\int_\Gamma (\mathbf{a}\,d\mathbf{s}) = U(3, 0, 4) - U(1, 2, 2)$$

$$= -\frac{1}{\sqrt{3^2+4^2}} + \frac{1}{\sqrt{1+2^2+2^2}} = \frac{2}{15}.$$

Thus, the required work done by the vector $\mathbf{a} = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$ is equal to 2/15.

There arises a question how to find out whether the vector **a** has a potential function in the given domain Ω. To this end, let us introduce some new notions.

Let us introduce the symbolic vector $\nabla = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ which is called the *Hamiltonian* operator.*

* Hamilton, William Rowan (1805-1865). Irish mathematician, astronomer, and physicist.

The vector

$$\operatorname{curl} \mathbf{a} = \nabla \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

is called the *curl* (rotational) *of the vector* **a**.

Thus, we may say that the curl of a vector **a** is equal to the vector product of the symbolic vector (Hamiltonian operator) and the vector **a**.

The following two theorems give an answer to the above question.

Theorem 2. *If a vector* **a** *has on* Ω *the potential function* U *having continuous second partial derivatives, then*

$$\operatorname{curl} \mathbf{a} = \mathbf{0}.$$

Indeed, by the conditions of the theorem,

$$\frac{\partial U}{\partial x} = P, \quad \frac{\partial U}{\partial y} = Q, \quad \frac{\partial U}{\partial z} = R.$$

Therefore

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = \frac{\partial^2 U}{\partial y \, \partial z} - \frac{\partial^2 U}{\partial z \, \partial y} = 0,$$

$$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = \frac{\partial^2 U}{\partial z \, \partial x} - \frac{\partial^2 U}{\partial x \, \partial z} = 0,$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial^2 U}{\partial x \, \partial y} - \frac{\partial^2 U}{\partial y \, \partial x} = 0$$

in Ω, which was to be proved.

The converse is also true, but generally speaking, for simply connected domains.

A domain Ω is said to be *simply connected* if any closed piecewise curve γ within it can be deformed continuously to a point of Ω without leaving the domain Ω.

In this definition it is sufficient to assume that curves γ are non-self-intersecting. Such curves are boundaries of the corresponding simply connected domains $\omega \subset \overline{\omega} \subset \Omega$.

An entire space or a ball without its boundary (spherical surface) may serve as examples of simply connected domains. On the other hand, the entire three-dimensional space with a straight line deleted is an example of a not simply connected or multiply connected domain.

Theorem 3. *If the domain* Ω *is simply connected and given in it is a vector* $\mathbf{a}$ *with continuously differentiable components for which*

$$\operatorname{curl} \mathbf{a} = 0,$$

then the vector $\mathbf{a}$ *has a potential function (potential) in* Ω.

In the three-dimensional case Theorem 3 follows from Stokes'* formula which will be proved in Sec. 3.15, in the two-dimensional (plane) case it follows from Green's** formula to be proved in Sec. 3.7.

In the plane case we consider the field of the vector

$$\mathbf{a} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} \quad ((x, y) \in \Omega),$$

where $P(x, y)$ and $Q(x, y)$ are continuous functions in the domain Ω of the plane.

The function $U(x, y)$ is said to be potential for the vector $\mathbf{a}$ in Ω if

$$\frac{\partial U}{\partial x} = P(x, y), \quad \frac{\partial U}{\partial y} = Q(x, y), \quad (x, y) \in \Omega.$$

The above facts are valid for the plane as well. It is only necessary to omit z everywhere and to assume that $R \equiv 0$.

In the plane case the definition of a simply connected domain is retained. We would like to notice that a plane (two-dimensional space) with a point deleted is not a simply connected domain.

Example 3. The vector $\mathbf{a}$ with the components

$$P(x, y) = -\frac{y}{x^2+y^2}, \quad Q(x, y) = \frac{x}{x^2+y^2}$$

has continuous partial derivatives in the domain G which is a plane with the zero point removed.

If the vector $\mathbf{a}$ is written in the form

$$\mathbf{a} = P\mathbf{i} + Q\mathbf{j} + 0 \cdot \mathbf{k},$$

* Stokes, George Gabriel (1819-1903). British physicist and mathematician.

** Green, George (1793-1841). English mathematician.

hence it is seen that

$$\operatorname{curl} \mathbf{a} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

It is readily verified that in the present case

$$\operatorname{curl} \mathbf{a} = \mathbf{0} \quad (\text{in } G).$$

The domain G (of the plane!) is not simply connected. It does not satisfy the conditions of Theorem 3 and the theorem itself, as we shall see, does not hold for it.

Indeed, the curve γ (a circle)

$$x = \cos \theta, \quad y = \sin \theta \quad (0 \leqslant \theta \leqslant 2\pi),$$

is, obviously, closed and belongs to G. The line integral of the vector $\mathbf{a}$ about γ is equal to

$$\int_\gamma (\mathbf{a}\, ds) = \int_\gamma \left(\frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy \right)$$

$$= \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta)\, d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

We see that there is a closed curve $\gamma \subset G$, about which the integral of $\mathbf{a}$ is not equal to zero. This shows, by Theorem 1, that in G there is no potential function for the vector $\mathbf{a}$ under consideration.

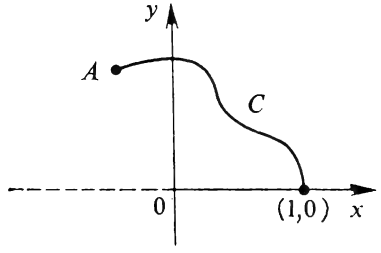

Fig. 75.

On the other hand, if the negative x-semiaxis is removed from the xy-plane or, as we say, if the plane is cut along the negative x-semiaxis, then the remaining set, which will be denoted by G_1 (Fig. 75), is simply connected, and since in G_1 curl $\mathbf{a} = \mathbf{0}$, then, by Theorem 3, the potential function of the vector $\mathbf{a}$ already exists in G_1. This function can be written in the following way:

$$U(A) = U(x, y) = \int_G \frac{-y\, dx + x\, dy}{x^2+y^2},$$

where $C \subset G_1$ is an oriented curve joining some initial fixed point A_0, say, (1, 0) to the variable point $A = (x, y) \in G_1$ (see Fig. 75).

Note that the curve C must not intersect the negative x-semiaxis.

Sec. 3.5. Exact Differential Equations

Consider the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0, \tag{1}$$

where M and N are continuous functions in some plane simply connected domain Ω.

We shall suppose that the left-hand member of (1) is a total differential, i.e. there exists in Ω a function $U(x, y)$ such that

$$\frac{\partial U}{\partial x} = M(x, y), \quad \frac{\partial U}{\partial y} = N(x, y). \tag{2}$$

In this case equation (1) is called an *exact differential equation*. It can be written in the form

$$dU(x, y) = 0. \tag{1'}$$

In order to solve equation (1), it is necessary to find the function $U(x, y)$ and equate it to an arbitrary constant

$$U(x, y) = C. \tag{3}$$

Equality (3) yields the total integral of equation (1) with respect to both forms of solutions: $y = y(x)$ and $x = x(y)$ (see Theorem 1 proved in Sec. 1.2).

Note that, by virtue of equalities (2), the function $U(x, y)$ is the potential of the vector

$$\mathbf{a} = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}, \quad (x, y) \in \Omega. \tag{4}$$

We know from Sec. 3.4 that to find the function U we have to compute the line integral

$$U(x, y) = \int_C (P\,dx + Q\,dy)$$

over a piecewise smooth curve $C \subset \Omega$, connecting a fixed point $A_0 = (x_0, y_0) \in \Omega$ with a variable point $A = (x, y) \in \Omega$.

We shall assume that Ω is a rectangle with sides parallel to x- and y-axes (Fig. 76).

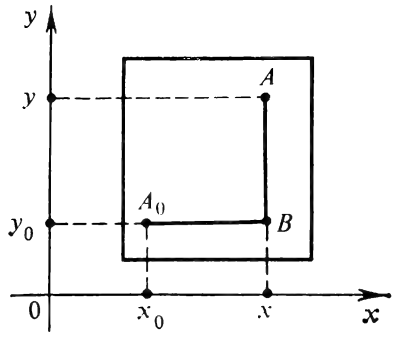

Fig. 76.

We can reach the point A by moving from the point A_0 along the path A_0BA. In this case we have to perform the following computations

$$U(x, y) = \int\limits_{C_{A_0B}} (P\,dx + Q\,dy) + \int\limits_{C_{BA}} (P\,dx + Q\,dy)$$

$$= \int\limits_{x_0}^{x} P(u, y_0)\,du + \int\limits_{y_0}^{y} Q(x, v)\,dv, \qquad (5)$$

since

$$\int\limits_{C_{A_0B}} Q\,dy = 0, \quad \int\limits_{C_{BA}} P\,dx = 0.$$

We recall here that from Theorems 2 and 3 of Sec. 3.4 it follows that for a simply connected plane domain Ω equation (1) will be an exact differential equation if and only if (curl $\mathbf{a} = \mathbf{0}$)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (6)$$

(of course, provided that M, N are continuously differentiable in Ω).

Example. Solve the differential equation

$$(x^{\alpha} + y^{\beta+1})dx + (\beta + 1)\, xy^{\beta}\, dy = 0, \tag{7}$$

where α, β are real numbers in the domain Ω of points (x, y) having positive coordinates $(x > 0, y > 0)$. Here

$$M(x, y) = x^{\alpha} + y^{\beta+1}, \quad N(x, y) = (\beta + 1)\, xy^{\beta},$$

$$\frac{\partial M}{\partial y} = (\beta + 1)\, y^{\beta}, \quad \frac{\partial N}{\partial x} = (\beta + 1)\, y^{\beta},$$

i.e.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ in } \Omega,$$

where Ω is a simply connected domain. Therefore

$$\operatorname{curl} \mathbf{a} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = \mathbf{0}$$

and, consequently, according to Theorem 3 of Sec. 3.4, there exists a function $U(x, y)$ whose total differential is the left-hand member of (7). This function can be obtained by formula (5):

$$U(x, y) = \int_{x_0}^{x} (u^{\alpha} + y_0^{\beta+1})\, du + \int_{y_0}^{y} (\beta + 1)\, xv^{\beta}\, dv$$

$$= \frac{x^{\alpha+1}}{\alpha+1} - \frac{x_0^{\alpha+1}}{\alpha+1} + xy_0^{\beta+1} - x_0 y_0^{\beta+1} + xy^{\beta+1} - xy_0^{\beta+1}$$

$$= \frac{x^{\alpha+1}}{\alpha+1} + xy^{\beta+1} + C \quad (\alpha \neq -1).$$

Thus, the general solution of equation (7) in Ω is the function $y(x)$ satisfying the equation

$$\frac{x^{\alpha+1}}{\alpha+1} + xy^{\beta+1} = C,$$

where C is an arbitrary constant. For $\alpha = -1$ the general solution is found from the equation

$$\ln |x| + xy^{\beta+1} = C.$$

Sec. 3.6. Orientation of a Plane Domain

There are two types of rectangular coordinate systems in the plane (see Figs. 77 and 78).

The difference in their arrangement consists in that it is impossible, by shifting these systems in the plane, to bring them to a position in which their positive x- and y-axes coincide.

Figure 77 represents a simply connected domain Ω with a piecewise smooth oriented boundary Γ. For the coordinate

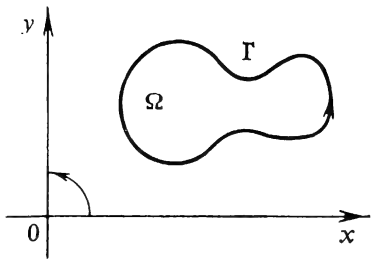

Fig. 77.

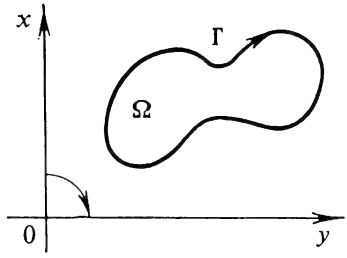

Fig. 78.

system taken in this figure the anticlockwise direction of traversal is regarded as positive. When Γ is traversed anticlockwise, the domain Ω remains on the left.

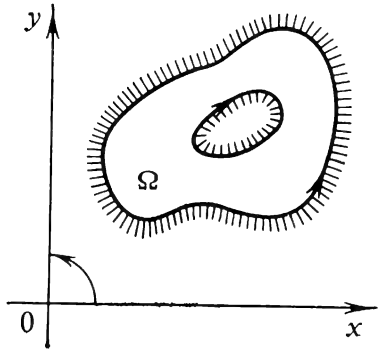

Fig. 79.

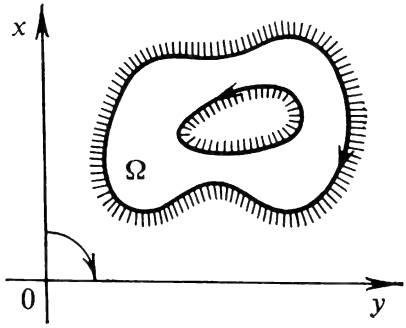

Fig. 80.

Figure 78 depicts the same domain Ω, but this time in another rectangular coordinate system. In this case the clockwise direction of traversal is regarded as positive and

when Γ is traversed clockwise, the domain Ω then remains on the right.

In the case of the coordinate system chosen in Fig. 79 the boundary (contour) Γ of a multiply (not simply) connected plane domain Ω is defined as positive or negative oriented depending on wether Ω remains on the left or on the right when Γ is traversed clockwise.

Figures 79 and 80 represent doubly connected domains Ω in different coordinate systems. The arrows on their boundaries indicate the positive direction of Γ.

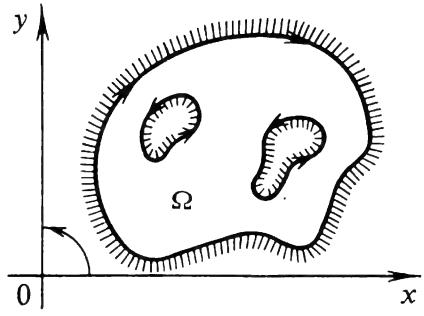

Fig. 81.

Shown in Fig. 81 is a triply connected domain with a negatively oriented boundary.

A domain Ω is said to be oriented positively or negatively if its boundary Γ is oriented positively or negatively.

If Ω is oriented positively, then Ω_- denotes the same domain, but oriented negatively.

The following agreement is useful: let Ω be a domain in the xy-plane, and let Ω_+ denote the domain Ω oriented positively and Ω_- the domain Ω oriented negatively. Then, by definition,

$$\iint\limits_{\Omega_+} f(x, y)\,dx\,dy = \iint\limits_{\Omega} f(x, y)\,dx\,dy,$$

$$\iint\limits_{\Omega_-} f(x, y)\,dx\,dy = -\iint\limits_{\Omega} f(x, y)\,dx\,dy,$$

where the right-hand members are ordinary double integrals of $f(x, y)$ over Ω.

Sec. 3.7. Green's Formula

For sufficiently general plane domains Ω with a positively oriented boundary Γ the following formula holds true:

$$\iint\limits_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int\limits_{\Gamma} (P\, dx + Q\, dy). \qquad (1)$$

This is *Green's formula* in which it is supposed that Q, $\frac{\partial Q}{\partial x}$, P, $\frac{\partial P}{\partial y}$ are continuous in the closure $\bar{\Omega}$ of the domain Ω.

To begin with, let us consider the plane domain Ω represented in Fig. 82 which will be called an *elementary H_y-domain*. From below and above this domain is bounded by piecewise smooth curves having the following respective equations:

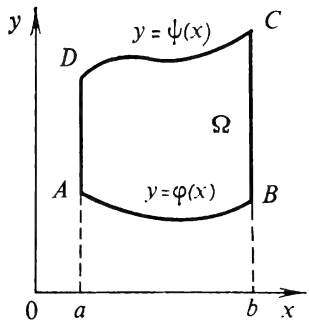

Fig. 82.

$$y = \varphi(x), \quad y = \psi(x),$$

$$\varphi(x) \leqslant \psi(x) \quad (a \leqslant x \leqslant b).$$

From the sides Ω is bounded by line segments parallel to the y-axis. Note that these line segments can degenerate into a point.

Thus, the boundary Γ of the domain Ω consists of four parts:

$$\Gamma = \Gamma_{CD} + \Gamma_{DA} + \Gamma_{AB} + \Gamma_{BC}.$$

For such domain Ω the following equality holds:

$$\iint\limits_{\Omega} \frac{\partial P}{\partial y}\, dx\, dy = \iint\limits_{\Omega} \frac{\partial P}{\partial y}\, dy\, dx = \int\limits_{a}^{b} dx \int\limits_{\varphi(x)}^{\psi(x)} \frac{\partial P}{\partial y}\, dy$$

$$= \int\limits_{a}^{b} [P[x, \psi(x)] - P[x, \varphi(x)]]\, dx$$

$$= -\int\limits_{b}^{a} P[x, \psi(x)]\, dx - \int\limits_{a}^{b} P[x, \varphi(x)]\, dx$$

$$= -\int_{\Gamma_{CD}} P(x,\ y)\,dx - \int_{\Gamma_{DA}} P(x,\ y)\,dx - \int_{\Gamma_{AB}} P(x,\ y)\,dx$$

$$- \int_{\Gamma_{BC}} P(x,\ y)\,dx = -\int_{\Gamma} P(x,\ y)\,dx.$$

Let us explain the last equality. The curve Γ_{CD} has the parametric equations (with the parameter x)

$$x = x,\quad y = \psi(x),$$

the point C corresponding to the value $x = b$, and the point D to the value $x = a$. The curve Γ_{AB} is defined by the equations

$$x = x,\quad y = \varphi(x).$$

Here to the value $x = a$ there corresponds the point A and to the value $x = b$ the point B. Finally, the line segment BC has the equations $x = b$, $y = y$ (with the parameter y). Along this segment $dx = 0$, therefore in reality

$$\int_{\Gamma_{BC}} P(x,\ y)\,dx = 0.$$

Analogously, the integral over the line segment DA is equal to zero:

$$\int_{\Gamma_{DA}} P(x,\ y)\,dx = 0.$$

Thus, we have proved that

$$\iint_{\Omega} \frac{\partial P}{\partial y}\,dx\,dy = -\int_{\Gamma} P(x,\ y)\,dx. \tag{2}$$

This formula is extended to any domain Ω which can be cut into a finite number of elementary H_y-domains. Such domain will be called simply H_y-domain.

Indeed, let $\Omega = \sum_{j=1}^{N} \Omega_j$, where Ω_j are elementary H_y-domains with a boundary γ_j, which will be regarded as positive-

ly oriented (Fig. 83). Then

$$\iint\limits_{\Omega} \frac{\partial P}{\partial y}\, dx\, dy = \sum_{j=1}^{N} \iint\limits_{\Omega_j} \frac{\partial P}{\partial y}\, dx\, dy = \sum_{j=1}^{N} - \int\limits_{\gamma_j} P\, dx$$

$$= - \int\limits_{\Gamma} P(x, y)\, dx. \quad (3)$$

We should like to explain the third equality. It is of importance to note here that if two contours γ_i and γ_j $(i \neq j)$

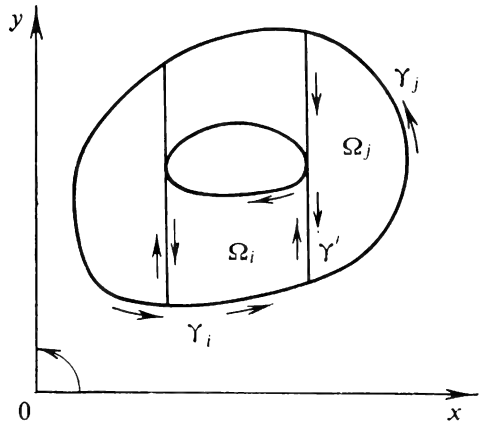

Fig. 83.

have a common piece γ', then the latter, being a part of both γ_i and γ_j is oriented oppositely, and therefore the line integrals of P over γ' in both cases differ only by sign and their sum is equal to zero. This taken into consideration, the sum $\sum\limits_{j=1}^{N}$ in (3) will be reduced to the sum of line integrals over the pieces γ_j belonging to Γ which is equal to the integral about the contour Γ.

By analogy, we can introduce the notion of an elementary H_x-domain. In this case Ω is bounded by the piecewise smooth curves

$$x = \lambda(y), \quad x = \mu(y)$$

$$\lambda(y) \leqslant \mu(y) \quad (c \leqslant y \leqslant d)$$

from the left and from the right, respectively (Fig. 84). From above and below Ω is bounded by segments of straight lines parallel to the x-axis.

We can also say that an elementary H_x-domain is defined in the same way as an elementary H_y-domain, only now the role of y is played by x.

For the elementary H_x-domain of Ω we obtain

$$\iint_{\Omega} \frac{\partial Q}{\partial x}\, dx\, dy = \int_c^d dy \int_{\lambda(y)}^{\mu(y)} \frac{\partial Q}{\partial x}\, dx$$

$$= \int_c^d [Q(\mu(y),\ y) - Q(\lambda(y),\ y)]\, dy$$

$$= \int_c^d Q(\mu(y),\ y)\, dy + \int_d^c Q(\lambda(y),\ y)\, dy = \int_{\Gamma_{BC}} Q(x,\ y)\, dy$$

$$+ \int_{\Gamma_{DA}} Q(x,\ y)\, dy = \int_{\Gamma_{BC}} Q\, dy + \int_{\Gamma_{CD}} Q\, dy + \int_{\Gamma_{DA}} Q\, dy$$

$$+ \int_{\Gamma_{AB}} Q\, dy = \int_{\Gamma} Q(x,\ y)\, dy, \quad (4)$$

since $dy = 0$ over the segments CD and AB, then also

$$\int_{\Gamma_{CD}} Q\, dy = \int_{\Gamma_{AB}} Q\, dy = 0.$$

Formula (4), is also extended to any domain which can be cut into a finite number of elementary H_x-domains. Such domain will be called simply H_x-domain.

Hence, we have proved the following statement:

Theorem 1. *If a domain Ω is simultaneously an H_x- and an H_y-domain, then for this domain Green's formula holds true.*

To prove this theorem, it is sufficient to subtract equality (3) from equality (4), both equalities being valid for the domain Ω possessing the indicated properties.

The domain

$$\Omega = \left\{ \begin{matrix} x^2 < y < 1 \\ -1 < x < 1 \end{matrix} \right\} = \left\{ \begin{matrix} -\sqrt{y} < x < \sqrt{y} \\ 0 < y < 1 \end{matrix} \right\}$$

and the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}<1$$

may serve as examples of domains which are simultaneously H_x- and H_y-domains. Moreover, these domains are elementary H_x- and H_y-domains.

Remark 1. It is possible to prove a more general assertion. If a domain Ω is bounded by an arbitrary closed piecewise

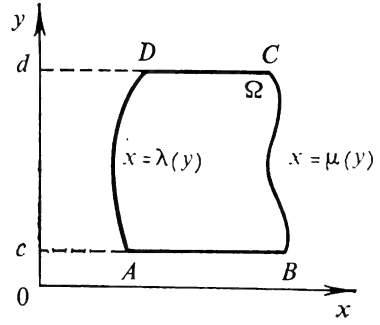

Fig. 84.

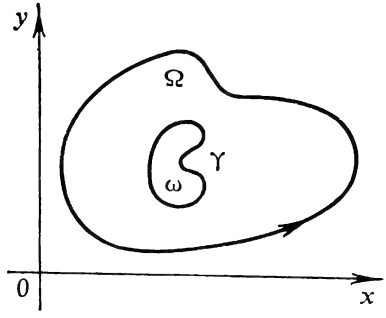

Fig. 85.

smooth non-self-intersecting contour Γ, then Green's formula (1) is valid for it.

Corollary 1. *If a plane domain Ω is simply connected and defined in it is a continuously differentiable vector*

$$\mathbf{a} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j},$$

for which

$$\operatorname{curl}\mathbf{a}=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}=\mathbf{0},$$

then $\mathbf{a}$ *has a potential in* Ω (i.e. there takes place the plane case of Theorem 3 proved in Sec. 3.4).

Indeed, let us take an arbitrary non-self-intersecting continuous piecewise smooth contour $\gamma\subset\Omega$, oriented positively (Fig. 85). It serves as a boundary of a certain domain ω. According to Green's theorem (formula) (see Remark 1)

$$\int_{\gamma}(P\,dx+Q\,dy)=\iint_{\omega}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\iint_{\omega}0\,dx\,dy=0,$$

and, since $\gamma \subset \Omega$ is an arbitrary closed non-self-intersecting contour, then, by Theorem 1 of Sec. 3.4, the vector **a** has a potential in Ω.

Remark 2. Since the double integral of a unit function over the domain Ω is equal to the area (measure) of the domain Ω, then, choosing the functions P and Q so that $\frac{dQ}{\partial x} - \frac{dP}{\partial y} \equiv 1$, we obtain different expressions for the area of the domain Ω in terms of the line integral

$$m\Omega = \int_\Gamma (P\,dx + Q\,dy).$$

In particular, for $P = -y/2$, $Q = x/2$, we get

$$m\Omega = \frac{1}{2} \int_\Gamma (-y\,dx + x\,dy).$$

Example. Compute the area bounded by the ellipse

$$x = a\cos t,\ y = b\sin t\ \ (0 \leqslant t \leqslant 2\pi).$$

According to formula (5), we have

$$S = \frac{1}{2} \int_\Gamma (-y\,dx + x\,dy)$$

$$= \frac{1}{2} \int_0^{2\pi} (ab\sin^2 t + ab\cos^2 t)\,dt = \frac{1}{2}\,ab \int_0^{2\pi} dt = \pi ab.$$

Sec. 3.8. Surface Integral of the First Type

Let a smooth surface S be defined by the equation

$$\mathbf{r}(u, v) = \varphi\mathbf{i} + \psi\mathbf{j} + \chi\mathbf{k}\ \ ((u, v) \in \Omega \rightleftarrows S,\ |\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v| > 0, \tag{1}$$

where Ω is a bounded domain with a piecewise smooth boundary and φ, ψ, χ are functions continuously differentiable in $\bar{\Omega}$. The symbol $\Omega \rightleftarrows S$ denotes one-to-one correspondence between the points $(u, v) \in \Omega$ and points of S.

Let, further, a continuous function $F(x, y, z)$ be defined in $\bar{S}$ or its neighbourhood. We then separate Ω into parts with piecewise smooth boundaries which do not intersect pairwise or intersect only along their boundaries. To each part Ω_j there corresponds a definite part S_j of the surface S. Let $A_j = (x_j, y_j, z_j)$ be an arbitrary point on S_j and let us form the sum

$$\Pi_N = \sum_{j=1}^{N} F(A_j) \mid S_j \mid,$$

where $\mid S_j \mid$ is the area of S_j (see Sec. 2.11). Its limit

$$\lim_{\max d(\Omega_j) \to 0} \sum_{j=1}^{N} F(A_j) \mid S_j \mid = \int_S F(x, y, z)\, dS \qquad (2)$$

is called the *integral of the function F through the surface S*, or the *surface integral of the first type*.

For instance, if a mass with density F is distributed on S, then the integral of F taken through S will express the total mass of S.

Integral (2) is computed by the following formula:

$$\int_S F(x, y, z)\, dS = \int_\Omega F[\varphi, \psi, \chi] \mid \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \mid du\, dv, \qquad (3)$$

where the right-hand member is an ordinary multiple integral with respect to $(u, v) \in \Omega$.

In particular, if a smooth surface S is defined by the equation $z = f(x, y)$ $((x, y) \in G)$, where f is continuous together with its first-order partial derivatives in $\bar{G}$, then we may regard that it is represented parametrically in terms of the parameters x, y:

$$x = x, \quad y = y, \quad z = f(x, y).$$

Then

$$\dot{\mathbf{r}}_x \times \dot{\mathbf{r}}_y = -f'_x \mathbf{i} - f'_y \mathbf{j} + \mathbf{k},$$

$$\left| \dot{\mathbf{r}}_x \times \dot{\mathbf{r}}_y \right| = \sqrt{1 + p^2 + q^2} \qquad \left(p = \frac{\partial f}{\partial x},\ q = \frac{\partial f}{\partial y} \right)$$

and, consequently

$$\int_S F(x, y, z)\, dS = \int_G F(x, y, f(x, y))\sqrt{1 + p^2 + q^2}\, dx\, dy. \tag{4}$$

Let us prove formula (3). Let $A_j = (x_j, y_j, z_j)$ and

$$x_j = \varphi(u_j, v_j),\ y_j = \psi(u_j, v_j),\ z_j = \chi(u_j, v_j),$$
$$(u_j, v_j) \in \Omega_j \quad (j = 1, \ldots, N).$$

Then

$$\Pi_N = \sum_{j=1}^{N} F(A_j) \int_{\Omega_j} \left| \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \right| du\, dv$$

$$= \sum_{j=1}^{N} F(A_j) \left| \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \right|_j^* | \Omega_j | = \sum_{j=1}^{N} F(A_j) \left| \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \right|_j | \Omega_j | + \varepsilon N$$

$$\to \int_{\Omega} F[\varphi, \psi, \chi] \left| \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \right| du\, dv \ (\max d(\Omega_j) \to 0),$$

where the symbol $|\ |_j$ means that substituted in $|\ |$ is a point A_j, and the symbol $|\ |_j^*$ that $|\ |$ contains a point such that the mean-value theorem is fulfilled for the integral

$$\int_{\Omega_j} \left| \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \right| du\, dv = \left| \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \right|_j^* | \Omega_j |$$

(see Sec. 2.3, Theorem 3).

It is obvious that $(K > | F(A) |)$ for any small η

$$| \varepsilon_N | = \left| \sum_{j=1}^{N} F(A_j) \left[\left| \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \right|_j^* - \left| \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \right|_j \right] | \Omega_j | \right|$$
$$\leqslant K\eta \sum_{j=1}^{N} | \Omega_j | = K\eta | \Omega |$$

if only $d(\Omega_j) < \delta$, where δ is a number depending on η, since the function $| \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v |$ is continuous on Ω.

Sec. 3.9. Orientation of a Surface

Consider the piece of a smooth surface S defined by the vector equation

$$\mathbf{r} = \mathbf{r}(u, v) = \varphi(u, v)\mathbf{i} + \psi(u, v)\,\mathbf{j} + \chi(u, v)\,\mathbf{k} \quad ((u, v) \in \bar{\Omega}), \qquad (1)$$

where φ, ψ, χ are functions continuously differentiable in the closure $\bar{\Omega}$ of a domain Ω with a piecewise smooth boundary and

$$|\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v| > 0, \quad (u, v) \in \bar{\Omega}. \qquad (2)$$

As ever, we assume that there takes place one-to-one correspondence $\bar{\Omega} \rightleftarrows S$ between the points $(u, v) \in \bar{\Omega}$ and points of S.

The unit normal at an arbitrary point of S is defined by the formula

$$\mathbf{n} = \pm \frac{\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v}{|\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v|}, \quad (u, v) \in \bar{\Omega}. \qquad (3)$$

To the sign "+" there corresponds one side of the surface S with a brush of unit normal vectors continuously depending on (u, v) and to the sign "—" the other side of S.

Let us give a definition: *If from every point A of a smooth surface S a unit normal $\boldsymbol{n}(A)$ can be drawn so that the obtained function of A is continuous throughout the entire surface S, then S is said to be an orientable surface.*

The function $\boldsymbol{n}(A)$ is also called a continuous field of normals.

A surface for which such function $\mathbf{n}(A)$ is defined is said to be oriented with the aid of $\mathbf{n}(A)$. If we say that S is an oriented surface, then by this we understand that S means not only a surface (a set of points), but also the fact that on it the indicated continuous (on S) function $\mathbf{n}(A)$ is defined. We also say that $\mathbf{n}(A)$ defines a definite side of an oriented smooth surface (where the brush of unit vectors $\mathbf{n}(A)$, continuously depending on A and emanating from S, is found).

The same surface, but oriented in an opposite way (by a brush of unit normal vectors going in the opposite direction), should be denoted by another symbol. Two such oppo-

sitely oriented surfaces are conveniently denoted by the symbols S_+ and S_-. One of the surfaces is arbitrarily designated by S_+, the other automatically receiving the designation S_-.

The xy-plane is a simplest example of an orientable surface. The unit vectors perpendicular to the plane and passing in the positive direction of the z-axis define one side of the plane, and the vectors in the negative direction of the z-axis the other side of the surface ($\mathbf{n}(A) = \pm \mathbf{k}$).

The surface of an ellipsoid is also orientable: a unit vector of the normal drawn from some of its points into the

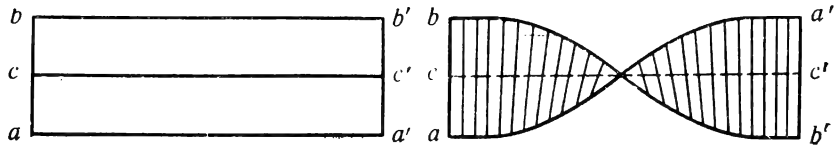

Fig. 86.

exterior of the ellipsoid, obviously, continues (identically!) over the entire surface. In this way the surface is oriented (i.e. the exterior side of the ellipsoid is defined). The opposite orientation of this surface is defined by the unit normal vector passing inside the ellipsoid (the interior of ellipsoid).

The aforegoing shows that if S is a smooth surface defined by parametric equations (1) with the above indicated properties, then it is orientable. In formula (3), to the plus sign there corresponds a certain orientation of S, the opposite orientation corresponding to the minus sign.

But in general, there exist *nonorientable* smooth surfaces.

If we take a long rectangular strip paper $abb'a'$ (Fig. 86) and paste its two ends ab and $a'b'$ together (so that the points a, b' and b, a' are pasted pairwise) after giving it half a twist, then we shall obtain a nonorientable surface called the *Möbius* strip* (see Fig. 87).

The line segment cc' marked in Fig. 86 is the midline of the rectangular strip of paper. On the Möbius strip to this line there corresponds a closed curve cc' whose points c and c'

* Möbius, August Ferdinand (1790-1868). German mathematician and astronomer.

have merged into one point. Let us now emanate from c a unit normal $\mathbf{n}\ (c)$ in an arbitrary but definite way. Once the direction $\mathbf{n}\ (c)$ is chosen (from two possible directions), this determines the choice of $\mathbf{n}\ (A)$ for all the points $A \in cc'$, if we want the vector $\mathbf{n}(A)$ to be continuously dependent on A.

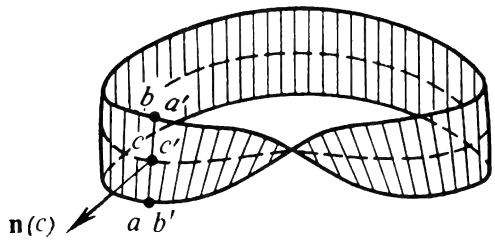

Fig. 87.

But at the point c' the vector $\mathbf{n}\ (c')$ is already chosen, since the points c and c' coincide. It is evident that if a point of the midline of the rectangle is continuously moved from c to c' then the unit normal $\mathbf{n}(A)$, where A is a point of the Möbius strip, will tend to $-n\ (c)$ but not to $\mathbf{n}\ (c)$, and, consequently, the vector function $\mathbf{n}\ (A)$ turns out to be discontinuous at the point $c = c' \in S$. Hence, the Möbius strip is nonorientable.

Sec. 3.10. Coordinate System and Orientation of a Surface

In the three-dimensional space there are two essentially different types of rectangular coordinates shown in Figs. 88 and 89. The distinction between these types is that none of the coordinate systems of the first type can be moved so that its origin O and the positive semiaxes Ox, Oy and Oz are brought into coincidence with the origin O and the positive semiaxes Ox, Oy, and Oz of a system belonging to the second type.

A coordinate system of the first type (Fig. 88) is called *right-handed*, while a system of the second type (Fig. 89) is called *left-handed**. The shortest rotation (in the xy-plane)

* See our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 11.

of the positive x-axis making it coincident with the positive y-axis when observed along the positive direction of the z-axis is seen in the clockwise direction in the case of Fig. 88 and in the anticlockwise direction in the case of Fig. 89.

It appears natural to attribute to either of the two coordinate systems under consideration a "screw" understood as an object consisting of the unit vector going in the positive

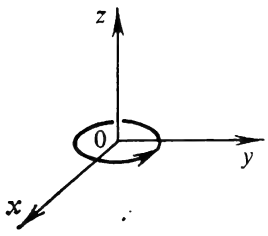

Fig. 88.

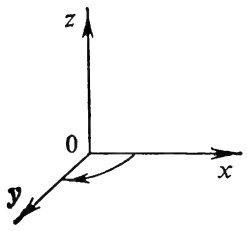

Fig. 89.

direction of the z-axis and of a circle (the "screw head") whose plane is perpendicular to the z-axis and whose boundary is oriented in the direction of the shortest rotation from the positive x-axis to the positive y-axis.

If the z-axis is regarded as the "screw axis" and the indicated circle as the head of a (real) right-hand screw (connected rigidly to its head), then if the head is rotated in the direction of the arrow shown in Fig. 88, the screw is driven in the positive direction of the z-axis. In the case of Fig. 89 we have the same effect when Oz is regarded as the axis of a left-hand screw.

The head of the screw (understood in this sense) can have a "warped" shape in the sense that it can be a part of a smooth surface, not necessarily plane, but such that the z-axis is the normal to the surface at the point O. In this case the object consisting of such a screw head with a chosen direction of traversing its boundary and of the unit normal vector is also regarded as a (right-hand or left-hand) screw associated with the given type of rectangular coordinates.

Finally, we can speak of such a (right-hand or left-hand) screw whose normal vector goes in an arbitrary direction, not necessarily coinciding with the z-axis. For our further aims the following construction is useful. Let us consider the

three-dimensional space with a chosen (right-handed or left-handed) rectangular coordinate system in which an oriented surface S is specified. This means that it is possible to draw a unit normal vector $\mathbf{n}(P)$ at each point $P \in S$ so that $\mathbf{n}(P)$ is a continuous function of P. A ball $V(P)$ of a sufficiently small radius with centre at P cuts out of the surface S a connected part $\sigma(P)$ containing point P. Let us choose the direction of traversing the contour (the edge) $\sigma(P)$ of that part so that the vector $\mathbf{n}(P)$ and the surface element $\sigma(P)$

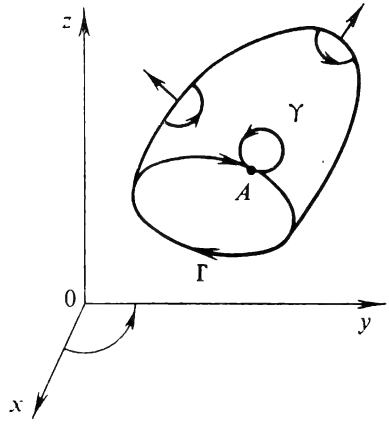

Fig. 90.

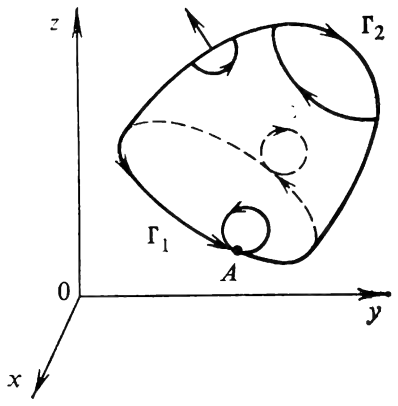

Fig. 91.

form a screw whose orientation is coherent with that of the coordinate system taken, that is let the screw be right-handed or left-handed depending on whether the coordinate system is right-handed or left-handed.

If the surface S has an edge Γ, then the above created construction leads in a natural way to a definite direction of traversing Γ (see Fig. 90). For instance, consider the point A belonging to the contour Γ. At this point we observe a coincidence of the directions in which Γ and the boundary γ of a small portion of the surface S adjoining the point A are traversed.

If the given surface were oppositely oriented, while the coordinate system remained the same, the directions of traversing Γ and the contours we have spoken of should be replaced by the opposite ones.

Figure 91 depicts an oriented surface with an edge consisting of two closed smooth curves Γ_1 and Γ_2.

Note also the following fact. Let an oriented smooth surface S be cut into two surfaces S_1 and S_2 by a smooth arch h (Fig. 92), the orientations of S_1 and S_2 being coherent with that of S. Then along the arc h the boundaries of S_1 and S_2 are described in the opposite directions. This remark

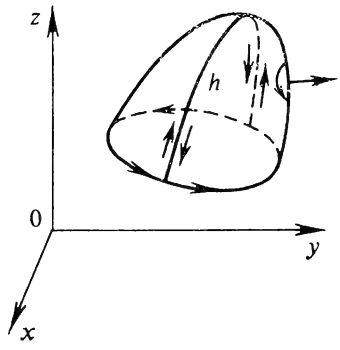

Fig. 92.

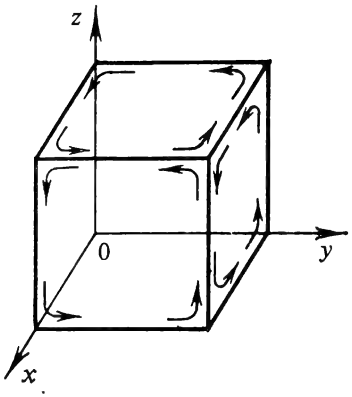

Fig. 93.

will be essential for the proper understanding of the notion of an oriented piecewise smooth surface to whose definition we now proceed.

A piecewise smooth surface S is said to be oriented if each of its smooth elements is oriented and if the corresponding directions in which the contours of the elements are described are coherent in the sense that they are opposite to each other along every common arc of any two such contours.

As an instance, see the cube shown in Fig. 93 whose boundary surface is oriented with the aid of its outer normal.

It is convenient to consider small elements of an oriented surface as vectors.

Let S be an oriented smooth surface. This means that at each point $A \in S$ a unit normal $\mathbf{n}(A)$ to S is drawn so that $\mathbf{n}(A)$ is a continuous function of A. Let σ be a small smooth element of S. To represent σ as a vector, we construct the vector $\boldsymbol{\sigma}$ whose length is equal to the area $|\sigma|$ of σ and whose direction coincides with that of the vector $\mathbf{n}(A)$,

where A is an arbitrary point belonging to σ. Thus, $\sigma = |\sigma|\,\mathbf{n}(A)$.

Of course, this construction does not define the vector σ uniquely. But if the diameter $d(\sigma)$ of σ is small, then the vectors $\mathbf{n}(A)$ $(A \in \sigma)$ lie within a small cone, and if σ is a variable small element containing permanently a fixed point A_0, then, obviously, $\mathbf{n}(A) \to \mathbf{n}(A_0)$ $(d(\sigma) \to 0)$, independently of the choice of the point $A \in \sigma$ for each σ.

By the vector of the differential of area of the surface S at its point $A \in S$ is naturally meant the vector $d\mathbf{S} = \mathbf{n}(A)\,dS$, which is equal to the product of the differential of area of S at the point A by the unit normal vector $\mathbf{n}(A)$ specifying the orientation of S.

If the surface S is represented by an equation

$$\mathbf{r} = \mathbf{r}(u, v) = \varphi\mathbf{i} + \psi\mathbf{j} + \chi\mathbf{k} \quad ((u, v) \in \bar{G}),$$

then $\mathbf{n}(A)$ is defined by one of the two equalities

$$\mathbf{n}(A) = \pm \frac{\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v}{|\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v|} \tag{1}$$

and

$dS = |\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v|\,du\,dv$. Hence

$$d\mathbf{S} = \pm (\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v)\,du\,dv. \tag{2}$$

For the sake of definiteness, in what follows we shall choose the plus sign in formulas (1) and (2). This can always be achieved by interchanging, if necessary, the roles of the parameters u and v. We can thus assert that *if we are given a definitely oriented smooth surface S, then it can always be represented by a vector function $\mathbf{r} = \mathbf{r}(u, v)$ such that the unit normal $\mathbf{n}(A)$ $(A \in S)$ specifying the orientation of S is expressed by the equality*

$$\mathbf{n}(A) = \frac{\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v}{|\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v|} \tag{3}$$

and, accordingly,

$$d\mathbf{S} = (\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v)\,du\,dv. \tag{4}$$

For the transformation from the old parameters (u, v) to some new parameters (u', v') not to lead to the appearance

of the minus sign in the above formulas, it is sufficient to require that the Jacobian $\frac{D(u, v)}{D(u', v')}$ of the transformation should be positive. Indeed, this follows from the relation

$$\mathbf{n}(A)\frac{\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v}{|\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v|} = \frac{\frac{D(y, z)}{D(u, v)}\mathbf{i} + \frac{D(z, x)}{D(u, v)}\mathbf{j} + \frac{D(x, y)}{D(u, v)}\mathbf{k}}{\sqrt{\left(\frac{D(y, z)}{D(u, v)}\right)^2 + \left(\frac{D(z, x)}{D(u, v)}\right)^2 + \left(\frac{D(x, y)}{D(u, v)}\right)^2}}$$

$$= \frac{\frac{D(y, z)}{D(u', v')}\mathbf{i} + \frac{D(z, x)}{D(u', v')}\mathbf{j} + \frac{D(x, y)}{D(u', v')}\mathbf{k}}{\sqrt{\left(\frac{D(y, z)}{D(u', v')}\right)^2 + \left(\frac{D(z, x)}{D(u', v')}\right)^2 + \left(\frac{D(x, y)}{D(u', v')}\right)^2}} \cdot \frac{\frac{D(u', v')}{D(u, v)}}{\left|\frac{D(u', v')}{D(u, v)}\right|}$$

$$= \frac{\dot{\mathbf{r}}_{u'} \times \dot{\mathbf{r}}_{v'}}{|\dot{\mathbf{r}}_{u'} \times \dot{\mathbf{r}}_{v'}|} \cdot \operatorname{sgn} \frac{D(u', v')}{D(u, v)}.$$

Thus, formula (3) (with the plus sign) for unit normal $\mathbf{n}(A)$ and, together with it, formula (4) are only invariant with respect to the transformations of the parameters having positive Jacobian. Therefore it is advisable to use the transformations of that type.

However, there are problems in which we have to consider transformations with negative Jacobians; then we should carefully choose the proper signs in the formulas.

There is no sense to speak of an arbitrary surface oriented in the space x, y, z that it is oriented positively or negatively. Quite another matter if a surface is plane and belongs to one of the coordinate planes.

An oriented domain G belonging to the xy-plane is called *positive* (*negative*) and is denoted by the symbol G_+ (G_-) if the unit normal, corresponding to the domain G, $\mathbf{n}(A) = \mathbf{k}$, where $\mathbf{k}$ is the unit vector of the z-axis.

This definition conforms to the definition given in Sec. 3.6. We should only assume there that we look at the xy-plane from the side of positive z's.

If in this definition x, y are replaced by y, z or z, x and also $\mathbf{k}$ is replaced by the unit vector $\mathbf{i}$ of the x-axis or by

* See our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 8.18.

the unit vector **j** of the y-axis; then we obtain the definition of G_+ for domains belonging to the yz- and zx-planes.

For domains G_- belonging to the xy-, or yz-, or zx-planes **k**, **i**, **j** should be replaced by $-\mathbf{k}$, $-\mathbf{i}$, $-\mathbf{j}$, respectively.

Sec. 3.11. Integral over an Oriented Domain in the Plane

In Sec. 3.6 we introduced the notion of the integral of f over an oriented domain. Namely,

$$\int\limits_{G_+} f\,dx\,dy = \int\limits_{G} f\,dx\,dy = -\int\limits_{G_-} f\,dx\,dy.$$

The expedience of this definition can be demonstrated by the following fact. Let us take two planes with rectangular coordinates x, y and x', y' having the same orientation. Let G be an oriented domain with piecewise smooth (oriented) boundary Γ lying in the xy-plane and let a continuously differentiable transformation

$$x' = \varphi(x, y),\ y' = \psi(x, y)\ ((x, y) \in \bar{G}) \qquad (1)$$

specify a one-to-one mapping of the domain G onto a domain G' in the $x'y'$-plane and of the boundary Γ of G onto the boundary Γ' of the domain G'. We shall suppose that the Jacobian of transformation (1) is different from zero:

$$D = \frac{D(x', y')}{D(x, y)} \neq 0 \quad (\text{in } G).$$

When the variable point (x, y) traverses Γ, transformation (1) induces a definite direction in which the corresponding point (x', y') describes Γ' and therefore G' can be regarded as a definitely oriented domain.

If $D > 0$ the contours Γ and Γ' are of the same orientation. If $D < 0$, then Γ and Γ' are oppositely oriented.

From what was said it follows that for any function $f(x, y)$ defined and continuous on the closure $\bar{G}$ of the oriented measurable domain G there holds the equality

$$\iint\limits_{G} f\,dx\,dy = \iint\limits_{G'} f \cdot \frac{D(x, y)}{D(x', y')}\,dx'\,dy',$$

where G' is the oriented domain in the $x'y'$-plane corresponding to G. In this formula expressing the rule for change of variables in a double integral over an oriented domain the Jacobian of the transformation is taken without the sign of absolute value.

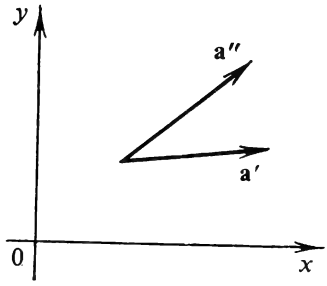

Fig. 94.

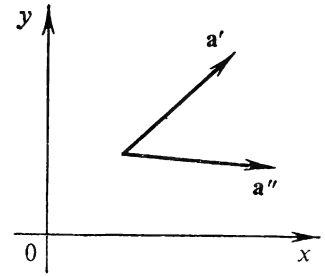

Fig. 95.

Let us also discuss the connection between the orientation of Γ and the sign of D. Let us take two noncollinear vectors $\mathbf{a}' = (a_1', a_2')$ and $\mathbf{a}'' (a_1'', a_2'')$ in the xy-plane. If the determinant

$$\Delta = \begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix}$$

is positive*, then the orientations of the pair $\mathbf{a}'$, $\mathbf{a}''$ and of the x- and y-axes are the same (Fig. 94). And if $\Delta < 0$, then their orientations are opposite (Fig. 95).

Transformation (1) specifies a mapping of the rectangular network of the coordinate lines in the xy-plane on a curvilinear one (see Figs. 96-98). There are two different characteristic classes of such mappings shown in Figs. 97 and 98.

As is seen in the figures, the square $ABCD$ goes into the curvilinear parallelogram $A'B'C'D'$, the vector $\overrightarrow{AB}$ goes, to within infinitesimals of higher order, into the tangent to the arc $\overset{\frown}{A'B'}$ at the point A' determined by the vector $\left(\frac{\partial x'}{\partial x}, \frac{\partial y'}{\partial x}\right)$ and the vector $\overrightarrow{AD}$ into the tangent to the arc $\overset{\frown}{A'D'}$ (at the point A') determined by the vector $\left(\frac{\partial x'}{\partial y}, \frac{\partial y'}{\partial y}\right)$. If the deter-

* See our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 12.

minant $D' = \frac{D(x', y')}{D(x, y)} > 0$, then the mutual disposition of these vectors is as shown in Fig. 97, and therefore the directions in which the variable points (x, y) and (x', y') cor-

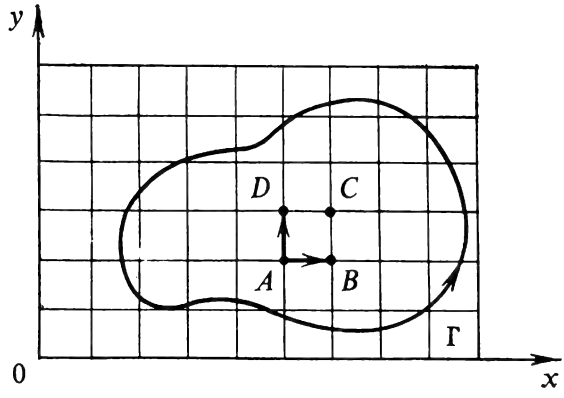

Fig. 96.

responding to each other describe the contours $ABCD$ and $A'B'C'D'$, respectively, are the same; hence the directions of traversing Γ and Γ' also coincide.

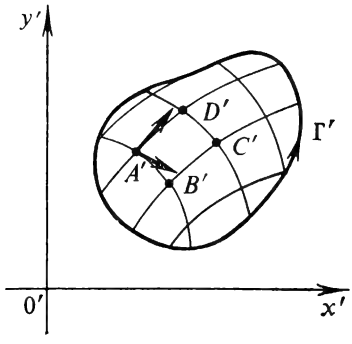

Fig. 97.

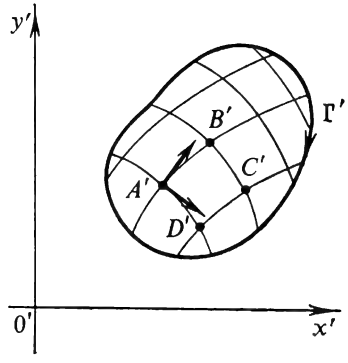

Fig. 98.

And if $D' < 0$ $(DD' \equiv 1)$, then the mutual orientation of the tangent vectors to $\widetilde{A'B'}$ and to $\widetilde{A'D'}$ changes to the opposite; therefore in this case Γ and Γ' are traversed in opposite directions (see Fig. 98).

The definition of the integrals over oriented domains G_+ and G_- lying in the other coordinate planes Oyz and Ozx are stated in like manner.

Sec. 3.12. Flux of a Vector Through an Oriented Surface

Let

$$\mathbf{a}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

be a continuous vector field defined in a domain H belonging to the three-dimensional space $E = E_3$ with rectangular coordinates x, y, z and let S^* be an oriented smooth surface lying in H and determined by a vector parametric equation

$$\mathbf{r} = \mathbf{r}(u, v) = \varphi\mathbf{i} + \psi\mathbf{j} + \chi\mathbf{k}, \ ((u, v) \in \Omega \mid \dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v \mid > 0), \tag{1}$$

where Ω is a measurable domain in the uv-plane and φ, ψ, χ are continuously differentiable functions on $\overline{\Omega}$.

We shall choose the field of unit normals to S^* specified by the vector equation

$$\mathbf{n}(A) = \frac{\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v}{|\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v|} \tag{2}$$

(in this connection see Sec. 3.10, (1), (3)). Then the direction cosines of $\mathbf{n}(A)$ are expressed by the equalities

$$\left.\begin{aligned} &\cos(\mathbf{n}, x) = \varkappa \frac{D(y, z)}{D(u, v)}, \quad \cos(\mathbf{n}, y) = \varkappa \frac{D(z, x)}{D(u, v)}, \\ &\cos(\mathbf{n}, z) = \varkappa \frac{D(x, y)}{D(u, v)}, \qquad \varkappa = 1/|\dot{\mathbf{r}}_u \times \dot{\mathbf{r}}_v|. \end{aligned}\right\} \tag{3}$$

The same surface considered without orientation will be denoted as S.

By the flux of the vector $\mathbf{a}$ *through the oriented surface* S^* *is meant the number*

$$\int_{S^*} (\mathbf{a}\, d\mathbf{S}^*) = \int_S (\mathbf{an})\, dS, \tag{4}$$

that is the value of the integral (of the first type) over the surface S of the scalar product

$$(\mathbf{an}) = P\cos(\mathbf{n}, x) + Q\cos(\mathbf{n}, y) + R\cos(\mathbf{n}, z)$$

of the vector **a** *of the field by unit normal* **n** *specifying the orientation of* S^*.

Since (**an**) is a continuous function of the point $A \in S$, the surface integral of the first type (the right-hand member of (4)) over S exists (this was proved in Sec. 3.8). The expression in the left-hand side of (4) is called the *surface integral of the second type**.

For instance, suppose that the vector field **a** in question defined in H is the velocity field of a stationary fluid flow, that is for each point $A \in H$ the vector **a** coincides with the velocity of the particle of fluid passing through that point, the velocity depending solely on A and being independent of time. Then the flux of the velocity vector **a** through an oriented surface S^* is equal to the volume of fluid flowing through S in unit time along the direction indicated by the normal $\mathbf{n}(A)$ specifying the orientation of S.

The following equality holds:

$$\int_{S^*} (\mathbf{an})\, dS = \int_{S^*} (P(\cos(\mathbf{n}, x) + Q\cos(\mathbf{n}, y) + R\cos(\mathbf{n}, z))\, dS$$

$$= \iint_{\Omega} \left(P\frac{D(y, z)}{D(u, v)} + Q\frac{D(z, x)}{D(u, v)} + R\frac{D(x, y)}{D(u, v)} \right) du\, dv, \qquad (5)$$

whose right-hand member is an ordinary double integral over the domain Ω; the functions P, Q and R under the sign of double integration are taken for the values of x, y, and z equal respectively to the functions φ, ψ, and χ dependent on u and v. This equality follows from (3) of this Section and formula (3) derived in Sec. 3.8.

It is often convenient to compute integral (5) in Cartesian coordinates. We shall demonstrate such calculations under the assumption that the smooth surface $\overline{S}$ in question can be projected in one-to-one manner on each of the coordinate

* For another designation of the integral of the second type see below (the right-hand member of (6) and (7)).

planes, the projections being measurable sets. Many smooth surfaces can be cut into a finite number of such pieces.

Thus, suppose that the smooth piece $\overline{S}$ can be described by each of the three functions

$$x = f_1(y, z) \quad ((y, z) \in \overline{S}_x),$$
$$y = f_2(z, x) \quad ((z, x) \in \overline{S}_y),$$
$$z = f_3(x, y) \quad ((x, y) \in \overline{S}_z),$$

continuous on the projections of $\overline{S}$ on the planes $x = 0$, $y = 0$, and $z = 0$, respectively, and possessing continuous partial derivatives, which, generally speaking, exist only in the measurable interiors S_x, S_y, and S_z of the projection (that is in the projections taken without their boundaries).

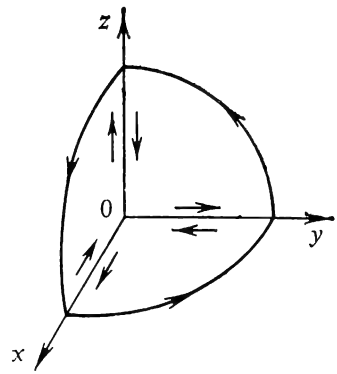

Fig. 99.

Let us denote as S_y^*, S_x^*, S_z^* the corresponding oriented projections of the oriented surface S^* on the planes $x = 0$, $y = 0$, and $z = 0$, respectively. The specification of the positive direction of traversing the contour of S^* (which determines the orientation of S^*) induces the corresponding directions in which the boundaries of the domains S_x, S_y and S_z are described (see Fig. 99). The cosine of the angle between the normal $\mathbf{n}$ to S and the positive z-axis is given by the formula

$$\cos(\mathbf{n}, z) = \pm \frac{1}{\sqrt{1+p^2+q^2}} \left(p = \frac{\partial f_3}{\partial x}, \quad q = \frac{\partial f_3}{\partial y}\right),$$

where the sign "+" or "—" is taken in accordance with the orientation of S^*. We have (see Sec. 3.8, (4) and Sec. 3.10)

$$\int_S R \cos(\mathbf{n}, z)\, dS$$
$$= \int_{S_z} R(x, y, f_3(x, y)) \frac{\pm 1}{\sqrt{1+p^2+q^2}} \sqrt{1+p^2+q^2}\, dx\, dy$$

$$= \pm \int\limits_{S_z} R(x, y, f_3(x, y))\, dx\, dy = \int\limits_{S_z^*} R(x, y, f_3(x, y))\, dx\, dy$$

$$= \int\limits_{S^*} R(x, y, z)\, dx\, dy, \qquad (6)$$

where the fourth integral is taken over the oriented domain S_z^* (see Sec. 3.8). As to the last integral in this chain, it should be regarded as the notation of the preceding one. This is the so-called *surface integral of the second type*. In order to evaluate this integral, we substitute $f_3(x, y)$ for z into the function $R(x, y, z)$ and take the integral of the resultant expression over the oriented projection S_z^*. We know from Sec. 3.8 that $\int\limits_{S_z^*} = \pm \int\limits_{S_z}$, where the sign "+" or "−" is taken depending on whether the domain S_z^* is oriented positively or negatively. Similar considerations apply to the other two integrals (Fig. 99):

$$\int\limits_S P \cos(\mathbf{n}, x)\, dS = \int\limits_{S_x^*} P(f_1(y, z), y, z)\, dy\, dz = \int\limits_{S^*} P(x, y, z)\, dy\, dz,$$

$$\int\limits_S Q \cos(\mathbf{n}, y)\, dS = \int\limits_{S_y^*} Q(x, f_2(z, x), z)\, dz\, dx = \int\limits_{S^*} Q(x, y, z)\, dz\, dx.$$

We have proved that the flux of the vector **a** through the oriented surface S^* specified by the normal **n** can be computed by the formula

$$\int\limits_S (\mathbf{an})\, dS$$

$$= \int\limits_{S^*} (P(x, y, z)\, dy\, dz + Q(x, y, z)\, dz\, dx + R(x, y, z)\, dx\, dy. \qquad (7)$$

If S^* is a surface which can be cut into a finite number of parts, $S^* = \Sigma S_k^*$ each of which can be projected in one-to-one manner on each of the three coordinate planes, then in order to compute the flux of **a** through S^*, we can find the

fluxes of **a** through each of these parts S_k^* as was shown above and then add up the results.

A spherical surface centred at the origin is naturally cut by the coordinate planes into eight pieces possessing the indicated property.

As we noted above, the right-hand member of (7) is called the surface integral of the second type.

Remark 1. If an orientable surface $S = G$ in the space (x_1, x_2, x_3) is a piece of a coordinate plane (its equation is $x_j = 0$ for some $j = 1, 2, 3$), then the flux of the vector **a** through G_+ is simply a double integral of the corresponding projection of the vector **a** on the corresponding axis. In particular, if G_+ is a part of the plane $x_3 = 0$ with the normal $\mathbf{n}(A) = +\mathbf{k}$, then $(\cos(n, z) = 1)$

$$\int_S (\mathbf{an})\, dS = \int_S R(x_1, x_2, 0)\, dx_1\, dx_2$$

$$= \int_{G_+} R(x_1, x_2, 0)\, dx_1\, dx_2 = \int_G R(x_1, x_2, 0)\, dx_1\, dx_2.$$

Conversely, if we are given, for instance, a double integral of the form

$$\int_G f(x, y)\, dx\, dy,$$

then it can be treated as the flux of the vector **a** through the domain G_+ whose projection on the z-axis is $R(x, y, z) = f(x, y)$.

Remark 2. The integral of the second type of the vector **a** over an oriented surface S^* changes when the orientation of the surface is reversed (namely, it changes sign).

Indeed, let S_-^* denote the same surface as S^*, but oriented oppositely. Then

$$\int_{S^*} (\mathbf{a}\, d\mathbf{S}^*) = \int_S (\mathbf{an})\, dS,$$

and

$$\int_{S_-^*} (\mathbf{a}\, d\mathbf{S}_-^*) = \int_S (\mathbf{a}, -\mathbf{n})\, dS = -\int_S (\mathbf{an})\, dS = -\int_{S^*} (\mathbf{a}\, d\mathbf{S}^*).$$

Sec. 3.13. Divergence. Gauss-Ostrogradsky* Theorem

Let E be the three-dimensional space with rectangular coordinates x, y, z and let $G \subset E$ be a domain with a piecewise smooth boundary S in which a vector field

$$\mathbf{a}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \quad ((x, y, z) \in G) \tag{1}$$

is defined. We shall suppose that the functions P, Q, R and their derivatives $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ are continuous in $\bar{G}$, whence it follows that for the vector $\mathbf{a}$ there exists the continuous function

$$\operatorname{div} \mathbf{a} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad ((x, y, z) \in \bar{G}), \tag{2}$$

which is called the *divergence of the vector* $\mathbf{a}$.

It is easy to see that

$$\operatorname{div} \mathbf{a} = \nabla \mathbf{a} \quad \left(\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\right),$$

i.e. the *divergence of the vector* $\mathbf{a}$ *is equal to the scalar product of the symbolic vector* ∇ (*Hamiltonian operator*) (see Sec. 3.4) *and the vector* $\mathbf{a}$.

We shall assume that the surface S is oriented with the aid of the unit normal $\mathbf{n}$ directed to the exterior of G.

The aim of this section is to prove the formula

$$\int_G \operatorname{div} \mathbf{a}\, dG = \int_S (\mathbf{an})\, dS \tag{3}$$

which holds under some additional conditions imposed on G. This equality is called the *Gauss-Ostrogradsky formula* (also known as the divergence theorem).

The Gauss-Ostrogradsky theorem reads: *the volume (triple) integral of the divergence of the vector* $\mathbf{a}$ *over the domain* G *is equal to the flux of that vector through the boundary* S *of the domain oriented with the aid of its outer normal.*

---

* Gauss, Carl Friedrich (1777-1855). Outstanding German mathematician.

** Ostrogradsky, Mikhail Vasilievich (1801-1861). Prominent Russian mathematician.

To begin with, let us consider the domain Λ shown in Fig. 100. A domain of this type will be referred to as an *elementary H_z-domain*. The upper and lower boundaries of Λ are the surfaces σ_1 and σ_2 with piecewise smooth boundary curves determined by the equations

$$z = \lambda_1(x, y), \quad z = \lambda_2(x, y), \quad \lambda_1(x, y) \leqslant \lambda_2(x, y)$$
$$((x, y) \in \bar{\Lambda}_z),$$

respectively, where Λ_z is a domain in the xy-plane with a piecewise smooth boundary γ and the functions λ_1 and λ_2

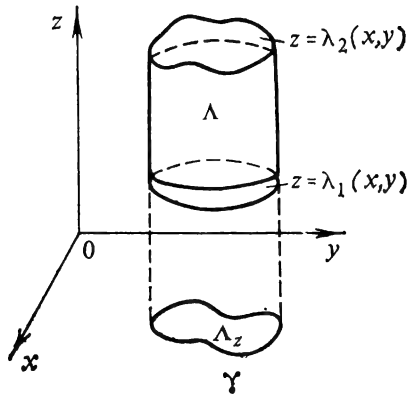

Fig. 100.

are continuous on $\bar{\Lambda}_z$ and have continuous partial derivatives in the open set Λ_z. The lateral boundary of Λ is the cylindrical surface σ^* with the curve γ as directrix and generatrix parallel to the z-axis.

Let S^* be the boundary of Λ oriented with the aid of the outer normal (see the explanations below). This induces the corresponding orientations of the surface elements σ_1^* and σ_2^* and the lateral surface σ^* of the domain Λ. For the domain Λ there hold the equalities (the explanations are given below)

$$\int_{\Lambda} \frac{\partial R}{\partial z}\, d\Lambda = \int_{\Lambda_z} \int dx\, dy \int_{\lambda_1(x, y)}^{\lambda_2(x, y)} \frac{\partial R}{\partial z}\, dz$$
$$= \int_{\Lambda} \int \{R(x, y, \lambda_2(x, y)) - R(x, y, \lambda_1(x, y))\}\, dx\, dy$$

$$= \iint\limits_{\sigma^*_{2,z}} R(x, y, \lambda_2(x, y))\, dx\, dy + \iint\limits_{\sigma^*_{1,z}} R(x, y, \lambda_1(x, y))\, dx\, dy$$

$$= \iint\limits_{S^*} R(x, y, z)\, dx\, dy. \qquad (4)$$

The normal $\mathbf{n}$ to σ_1^* forms an obtuse angle with the positive z-axis, while the normal to σ_2^* forms an acute angle with it; therefore the projections $\sigma_{1,z}^*$ of the element σ_1^* on the plane $z = 0$ is oriented negatively, while the projection $\sigma_{2,z}^*$ of σ_2^* is oriented positively. This accounts for the passage from the third term in (4) to the fourth. To the sum of the two integrals forming the fourth term we can formally add the integral

$$\iint\limits_{\sigma^*} R(x, y, z)\, dx\, dy = 0,$$

which is equal to zero, since $\cos(n, z) = 0$ for σ^*. Thus, the resultant sum of the three integrals is equal to the last integral in (4), that is to the flux of the vector $(0, 0, R)$ through S^*.

We have thus proved the Gauss-Ostrogradsky theorem for an elementary H_z-domain and for the vector of the form $(0, 0, R)$.

Further, let us call a domain G an H_z-domain if its closure $\overline{G}$ can be cut into a finite number of elementary H_z-domains

$$\overline{G} = \sum_{k=1}^{N} \overline{G}_k$$

so that the upper and lower parts of the boundaries of $\overline{G}_k$'s are some parts of the oriented boundary S^* of the domain G. We shall prove that for such a domain G and for a vector of the form $(0, 0, R)$ the Gauss-Ostrogradsky theorem also holds.

To this end, let us denote by $S_{1,k}$ and $S_{2,k}$ the lower and upper parts of the boundaries of $\overline{G}_k$'s respectively, and by S_k the lateral parts of the boundaries of G_k's. Then we have

(see the explanations below)

$$\int\limits_G \frac{\partial R}{\partial z}\, dG = \sum_{k=1}^{N} \int\limits_{G_k} \frac{\partial R}{\partial z}\, dG$$

$$= \sum_{k=1}^{N} \left(\int\limits_{S^*_{1,k}} R(x, y, z)\, dx\, dy + \int\limits_{S^*_{2,k}} R(x, y, z)\, dx\, dy \right.$$

$$\left. + \int\limits_{s^*_k} R(x, y, z)\, dx\, dy \right) = \int\limits_{S^*} R(x, y, z)\, dx\, dy,$$

since the integrals over S^*_k's are, obviously, equal to zero, while for the parts $S^*_{1,k}$ and $S^*_{2,k}$ we can assert that either their union coincides with the surface S^* or, if otherwise, the set

$$\sigma = S^* - \sum_1^N S^*_{1,k} - \sum_1^N S^*_{2,k}$$

is a part of S^* such that the normal at its every point is perpendicular to the z-axis and therefore the integral over σ is equal to zero.

By analogy with the above, we can define the notions of H_x-domains and H_y-domains. For instance, an H_x-domain possesses the property that its closure can be put into a finite number of closures of elementary H_x-domains, the latter being understood in the same sense as the elementary H_z-domains with the only difference that the role of the z-axis is played by the x-axis. Similar considerations show that for an H_x-domain G there holds the equality

$$\int\limits_G \frac{\partial P}{\partial x}\, dG = \int\limits_{S^*}\int P(x, y, z)\, dy\, dz,$$

that is the Gauss-Ostrogradsky formula holds for the H_x-domains and for the vectors of the form $(P, 0, 0)$. Analogously, for an H_y-domain G there holds the formula

$$\int\limits_G \frac{\partial Q}{\partial y}\, dG = \int\limits_{S^*}\int Q(x, y, z)\, dz\, dx,$$

expressing the Gauss-Ostrogradsky theorem for a vector $(0, Q, 0)$.

Now let us suppose that a given domain G is simultaneously an H_x-, an H_y-, and an H_z-domain; then, obviously, the Gauss-Ostrogradsky theorem holds for any continuously differentiable vector field $\mathbf{a} = (P, Q, R)$ defined on $\bar{G}$, that is we have the equality

$$\iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz$$

$$= \iint_{S^*} (P(x, y, z)\, dy\, dz + Q(x, y, z)\, dz\, dx + R(x, y, z)\, dx\, dy), \tag{5}$$

where the integral on the right-hand side is taken over the surface S^* oriented with the aid of the outer normal to G.

On setting $P = x$, $Q = y$, and $R = z$ in the Gauss-Ostrogradsky formula, we obtain the formula

$$|G| = \frac{1}{3} \iint_{S^*} (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

expressing the volume of the domain G in terms of the surface integral taken over its oriented (with the aid of the outer normal) boundary S^*.

Practically, we usually deal with domains which are simultaneously H_x-, H_y- and H_z-domains.

Example 1. The ball $x^2 + y^2 + z^2 \leqslant 1$ is an H_z-domain and even an elementary H_z-domain, since its interior is bounded above and below by the smooth surfaces

$$z_1 = \sqrt{1 - x^2 - y^2}, \text{ and } z_2 = -\sqrt{1 - x^2 - y^2},$$

where the functions z_1 and z_2 are continuous in the closed circle $x^2 + y^2 \leqslant 1$, whose boundary curve is smooth. It is evident that the ball under consideration is also an H_x- and H_y-domain.

Example 2. Let us consider the torus T obtained by the rotation of the circle $(x - b)^2 + y^2 = a^2$ $(0 < a < b)$ lying in the xy-plane about the y-axis. To show that T is an H_y-domain it suffices to cut the boundary surface of T into two parts with the aid of the xz-plane. Further, the planes $x = b - a$ and $x = a - b$ divide T into four elementary H_z-domains, while the planes $z = b - a$ and $z = a - b$ di-

vide it into four elementary H_x-domains (Fig. 101 shows one half of the torus T whose equation in Cartesian coordinates has the form $(\sqrt{x^2 + z^2} - b)^2 + y^2 = a^2$).

The Gauss-Ostrogradsky formula transforms a volume integral over a domain into a surface integral taken over its boundary.

To elucidate the physical meaning of the notion of the divergence of a vector, let us suppose that the given domain

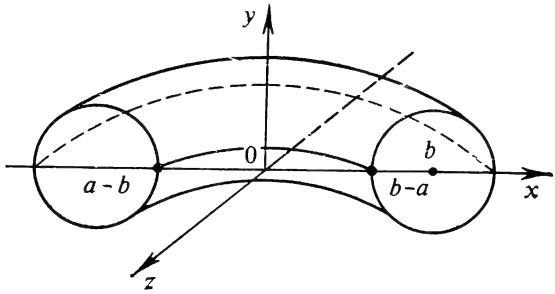

Fig. 101.

G is occupied by a stationary flow of a fluid so that the velocity of the particles of fluid passing through an arbitrary point (x, y, z) is equal to $\mathbf{a} = \mathbf{a}(x, y, z)$. Let us take an arbitrary but fixed point $A = (x, y, z) \in G$ and embed it in a ball $V_\varepsilon \subset G$ of radius $\varepsilon > 0$. Let S_ε^* be the boundary of the ball (the spherical surface) oriented with the aid of the outer normal. Then, by the Gauss-Ostrogradsky formula, we have

$$\iint_{S_\varepsilon} (\mathbf{a}\, d\mathbf{S}) = \iiint_{V_\varepsilon} \operatorname{div} \mathbf{a}\, dx\, dy\, dz.$$

The left-hand member of this equality expresses the volume of the fluid flowing out of V_ε (to the exterior of S_ε) in unit time. Applying the mean-value theorem to its right-hand member, we get

$$\iint_{S_\varepsilon} (\mathbf{a}\, d\mathbf{S}) = |V_\varepsilon| \operatorname{div} \mathbf{a}_1, \tag{6}$$

where $|V_\varepsilon|$ is the volume of V_ε and $\mathbf{a}_1$ is the velocity of the fluid at some point belonging to V_ε. Let us divide both

members of the obtained equality by $|V_\varepsilon|$ and pass to the limit as $\varepsilon \to 0$; by the continuity of div **a**, this limit exists and is equal to the divergence of **a** at the chosen $A = (x, y, z)$:

$$\operatorname{div} \mathbf{a} = \lim_{\varepsilon \to 0} \frac{1}{|V_\varepsilon|} \int\int_{S_\varepsilon} (\mathbf{a}\, d\mathbf{S}). \tag{7}$$

Thus, div **a** is equal to the value of the intensity of sources of fluid (continuously distributed over G) at the point $A = (x, y, z)$. If div **a** $= 0$ at the point A (or throughout G), then the intensity of sources is equal to zero at A (or throughout G). If div **a** < 0, then at the corresponding point there is in fact not a source but a sink of fluid.

From the above physical considerations it appears evident that div **a** is an invariant with respect to all transformations of the rectangular coordinates. The same conclusion can be drawn on the basis of purely mathematical considerations.

As we know (see our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 18), the scalar product of vectors is an invariant with respect to any transformations of the coordinates, therefore the divergence (equal to the scalar product of the symbolic vector ∇ and the vector **a**) is also an invariant with respect to all transformations of the rectangular coordinates. Of course, we suppose (by definition) that the coordinates of the symbolic vector $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ are transformed by the same formulas as the coordinates of ordinary vectors. More precisely, if the formulas for transforming the coordinates (x_1, x_2, x_3) of a point (vector) in one coordinate system to its coordinates (y_1, y_2, y_3) in another system have the form

$$y_l = \sum_{s=1}^{3} \alpha_{ls} x_s \quad (l = 1, 2, 3), \tag{8}$$

where $\alpha = (\alpha_{kl})$ is the corresponding orthogonal matrix, then

$$\frac{\partial}{\partial y_l} = \sum_{s=1}^{3} \alpha_{ls} \frac{\partial}{\partial x_s} \quad (l = 1, 2, 3). \tag{9}$$

The Hamiltonian operator is applied to a differentiable function f. As a result, we obtain the vector

$$\nabla f = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right),$$

called, as we know, the *gradient of the function f*, the latter regarded here as a scalar. Thus, $\nabla f = \operatorname{grad} f$ is the product of the vector ∇ by the scalar f, the result being a vector.

In the coordinate system (y_1, y_2, y_3)

$$\nabla f = \operatorname{grad} f = \left(\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \frac{\partial f}{\partial y_3}\right) = \frac{\partial f}{\partial y_1}\mathbf{i}_1 + \frac{\partial f}{\partial y_2}\mathbf{j}_1 + \frac{\partial f}{\partial y_3}\mathbf{k}_1,$$

where $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$ are the unit vectors of the system (y_1, y_2, y_3). Here, by virtue of (9),

$$\frac{\partial f}{\partial y_l} = \left(\alpha_{l1}\frac{\partial}{\partial x_1} + \alpha_{l2}\frac{\partial}{\partial x_2} + \alpha_{l3}\frac{\partial}{\partial x_3}\right) f = \sum_{s=1}^{3} \alpha_{ls}\frac{\partial f}{\partial x_s} \quad (l = 1, 2, 3). \tag{10}$$

The above formulas conform to the rules for differentiating the composite function $f(x_1, x_2, x_3)$ in which

$$x_s = \sum_{l=1}^{3} \alpha_{ls} y_l \quad (s = 1, 2, 3). \tag{11}$$

Formulas (11) are inverse to formulas (8) ($\alpha^{-1} = \alpha^*$, i.e. the coordinates x_s are expressed in terms of the coordinates y_k with the aid of column s of matrix α).

Here, we have obtained formulas (10), using only symbolic calculus.

Now, if one and the same vector field is defined in two rectangular coordinate systems (x_1, x_2, x_3) and (y_1, y_2, y_3) by the respective functions

$$\begin{aligned}\mathbf{a} &= a_1(x_1, x_2, x_3)\,\mathbf{i} + a_2(x_1, x_2, x_3)\,\mathbf{j} + a_3(x_1, x_2, x_3)\,\mathbf{k} \\ &= b_1(y_1, y_2, y_3)\,\mathbf{i}_1 + b_2(y_1, y_2, y_3)\,\mathbf{j}_1 \\ &\quad + b_3(y_1, y_2, y_3)\,\mathbf{k}_1,\end{aligned}$$

where the coordinates (b_1, b_2, b_3) are related with the coordinates (a_1, a_2, a_3) by formulas (8), (11), then at one and

the same point

$$\operatorname{div} \mathbf{a} = \sum_{l=1}^{3} \frac{\partial b_l}{\partial y_l} = \sum_{l=1}^{3} \left(\sum_{s=1}^{3} \alpha_{ls} \frac{\partial}{\partial x_s} \right) b_l$$

$$= \sum_{s=1}^{3} \frac{\partial}{\partial x_s} \left(\sum_{l=1}^{3} \alpha_{ls} b_l \right) = \sum_{s=1}^{3} \frac{\partial a_s}{\partial x_s}$$

Thus, we have proved once again the invariance of the divergence when transforming rectangular coordinates using the symbolic calculus only.

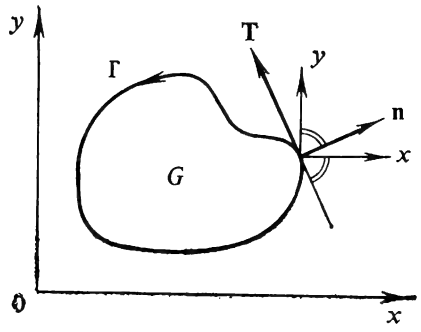

Fig. 102.

The Gauss-Ostrogradsky formula can also be written for the case when G is a domain in the xy-plane and

$$\mathbf{a}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j},$$

is a vector field defined in it. Denoting by $\mathbf{n}(A)$ the outer normal to the boundary Γ of the domain G ($A \in \Gamma$), which is supposed to be a piecewise smooth contour, we can write the equality

$$\iint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy = \int_\Gamma (\mathbf{an})\, ds,$$

where ds is the differential of arc length of the curve Γ.

If we agree that the direction of the tangent to the contour Γ coincides at every point of Γ with the positive direction of traversing Γ in which the arc length s is reckoned

along Γ, then (see Fig. 102)

$$\cos(\mathbf{n}, x) = \cos(\mathbf{T}, y) = \frac{dy}{ds},$$

$$\cos(\mathbf{n}, y) = \cos[\pi - (\mathbf{T}, x)] = -\cos(\mathbf{T}, x) = -\frac{dx}{ds}.$$

Therefore

$$(\mathbf{an})\, ds = P\, dy - Q\, dx,$$

$$\iint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\, dy = \int_\Gamma (P\, dy - Q\, dx).$$

On replacing P by Q and Q by $-P$ in the last formula, we arrive at Green's formula, which was already derived in Sec. 3.7.

Sec. 3.14. Solenoidal Field

The field (domain) Ω of the vector $\mathbf{a} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is called *solenoidal* (*tubular*) if the divergence of $\mathbf{a}$ in Ω is equal to zero:

$$\operatorname{div} \mathbf{a} = 0, \quad (x, y, z) \in \Omega.$$

By virtue of the Gauss-Ostrogradsky theorem, for a solenoidal field there takes place the equality

$$\iint_S (\mathbf{an})\, dS = \iiint_\omega \operatorname{div} \mathbf{a}\, dx\, dy\, dz = 0$$

for any closed oriented (in the exterior of ω) piecewise smooth surface S which is the boundary of the domain $\omega \subset \bar{\omega} \subset \Omega$, that is which is found strictly inside Ω.

In particular, if

$$S = S_1 + S_2,$$

as in Fig. 103,

$$\iint_{S_1} (\mathbf{an})\, dS + \iint_{S_2} (\mathbf{an})\, dS = 0$$

or

$$\iint_{S_1} (\mathbf{an})\, dS = \iint_{S_2^-} (\mathbf{an})\, dS,$$

where S_2^- is the same surface as S_2, but oriented in the opposite direction (towards the interior of ω).

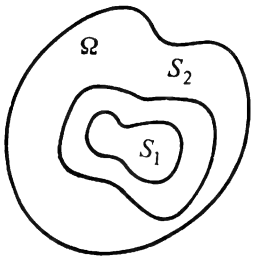

Fig. 103.

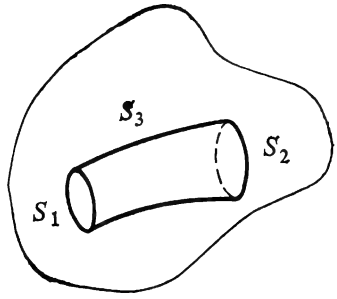

Fig. 104.

Consider in Ω the domain ω of a special form (Fig. 104)—a tube with the boundary S consisting of three smooth pieces:

$$S = S_1 + S_2 + S_3;$$

the vector **a** being tangent to S_3 at any point of S_3. Then

$$\iint\limits_{S_3} (\mathbf{an})\, dS = 0$$

and

$$\iint\limits_{S_1} (\mathbf{an})\, dS + \iint\limits_{S_2} (\mathbf{an})\, dS = 0$$

or

$$\iint\limits_{S_1} (\mathbf{an})\, dS = \iint\limits_{S_2^-} (\mathbf{an})\, dS.$$

We see that the flux of the vector **a** through S_2^- is equal to its flux through S_1, i.e. if, for example, **a** (x, y, z) is the velocity of the fluid flowing in Ω, then the amount of the fluid flowing into the tube in unit time is equal to the amount of the fluid flowing out of the tube.

Sec. 3.15. Stokes' Formula

Let us consider a continuously differentiable vector field

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

defined in a domain of the three-dimensional space E_3. In Sec. 3.4 we defined the notion of the curl of the vector $\mathbf{a}$:

$$\operatorname{curl}\ \mathbf{a} = \nabla \times \mathbf{a} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

As is known from vector algebra (see our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 18), the vector product of two vectors is an axial vector, i.e. it is invariant with respect to all the transformations of orthogonal coordinates preserving their orientation (in the sense that right-handed systems go into right-handed ones, while left-handed systems go into left-handed ones). Therefore curl $\mathbf{a} = \nabla \times \mathbf{a}$ is invariant with respect to the transformations of rectangular coordinates preserving their orientations. Consequently, we can assert without calculations that if our vector $\mathbf{a}$ has the components

$$\mathbf{a} = P_1(x', y', z')\,\mathbf{i}' + Q_1(x', y', z')\,\mathbf{j}' + R_1(x', y', z')\,\mathbf{k}',$$

in a new (also oriented) rectangular coordinate system, then there holds the equality

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \left(\frac{\partial R_1}{\partial y'} - \frac{\partial Q_1}{\partial z'}\right)\mathbf{i}' + \left(\frac{\partial P_1}{\partial z'} - \frac{\partial R_1}{\partial x'}\right)\mathbf{j}' + \left(\frac{\partial Q_1}{\partial x'} - \frac{\partial P_1}{\partial y'}\right)\mathbf{k}',$$

where $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ are the unit vectors of the coordinate axes of the system (x', y', z').

If an oriented contour Γ is closed, then the line integral of $\mathbf{a}$ taken along it will be called the circulation of the vector $\mathbf{a}$ over Γ and denoted by the symbol

$$\int_\Gamma (\mathbf{a}\,d\mathbf{l}) = \int_\Gamma (P\,dx + Q\,dy + R\,dz).$$

Here, $d\mathbf{l}$ is a vector going in the positive direction of the tangent to Γ whose length is equal to the differential of the arc length of Γ.

Our aim is to justify Stokes' formula

$$\iint_{S^*} (\operatorname{curl}\ \mathbf{a}\, d\mathbf{S}^*) = \iint_{S} (\mathbf{n}\ \operatorname{curl}\ \mathbf{a})\, dS = \int_{\Gamma} (\mathbf{a}\, d\mathbf{l}), \qquad (1)$$

expressing Stokes' theorem which states that the *flux of the vector* curl **a** *through an oriented surface* S^* *is equal to the circulation of the vector* **a** *over the coherently oriented contour* Γ *bounding the surface* S^*. Note that here S^* denotes the surface S oriented in some way.

We shall begin with the proof of Stokes theorem (i.e. of formula (1)) for a smooth surface element which can be projected in a one-to-one manner on each of the three coordinate planes.

Let us take an oriented surface element S^* with a piecewise smooth edge Γ representable in the following three ways:

$$z = f_1(x,\ y)\ (x,\ y) \in S_z,\ x = f_2(y,\ z)\ (y,\ z) \in S_x,$$

$$y = f_3(z,\ x)\ (z,\ x) \in S_y.$$

We thus suppose that each of these equations can be uniquely resolved with respect to any of the variables x, y, z and that the functions f_1, f_2, f_3 are continuously differentiable on the corresponding projections of S on the coordinate planes. We have

$$\iint_{S} (\mathbf{n}\ \operatorname{curl}\ \mathbf{a})\, dS = \iint_{S} \left\{ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos(\mathbf{n},\ x) \right.$$

$$\left. + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos(\mathbf{n},\ y) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos(\mathbf{n},\ z) \right\} dS. \qquad (2)$$

Let us consider the sum of those terms entering into the right-hand side of (2) which contain P. It can be expressed

(see the explanations below) as

$$-\int\int_S \left\{\frac{\partial P}{\partial y}\cos(\mathbf{n}, z)-\frac{\partial P}{\partial z}\cos(\mathbf{n}, y)\right\} dS$$

$$=-\int\int_S \left(\frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}\frac{\partial f_1}{\partial y}\right)\cos(\mathbf{n}, z)\, dS$$

$$=-\int\int_{S_z^*} \frac{\partial}{\partial y} P(x, y, f_1(x, y))\, dx\, dy$$

$$=\int_{\Gamma_z} P(x, y, f_1(x, y))\, dx=\int_{\Gamma} P(x, y, z)\, dx. \qquad (3)$$

Since grad $(f_1(x, y)-z)$ is orthogonal to the surface $z=f_1(x, y)$, the direction cosines of the normal at the point $(x, y, f_1(x, y))$ are proportional to the coordinates of the gradient:

$$\frac{\partial f_1}{\partial x}:\frac{\partial f_1}{\partial y}:(-1)=\cos(\mathbf{n}, x):\cos(\mathbf{n}, y):\cos(\mathbf{n}, z).$$

Hence

$$\cos(\mathbf{n}, y)=-\frac{\partial f_1}{\partial y}\cos(\mathbf{n}, z),$$

which implies the first equality in (3). The second equality follows from formula (6) of Sec. 3.12 and the rule for differentiating a composite function. The third equality follows from Green's formula. Finally, the last equality follows from the fact that the equations of the contour Γ have the form

$$x=\varphi(s),\ y=\psi(s),\ z=\chi(s)=f_1(\varphi(s), \psi(s)),$$

i.e. in the last two line integrals in (3)

$$dx=\varphi'(s)\, ds$$

and on the contour Γ_z

$$P(x, y, f_1(x, y))=P(\varphi(s), \psi(s),$$
$$f_1(\varphi(s), \psi(s)))=P(\varphi(s), \psi(s), \chi(s)),$$

which equals $P(x, y, z)$ on Γ.

The equalities

$$\iint_S \left(\frac{\partial Q}{\partial x} \cos(\mathbf{n}, z) - \frac{\partial Q}{\partial z} \cos(\mathbf{n}, x) \right) dS = \int_\Gamma Q(x, y, z)\, dy, \quad (4)$$

$$\iint_S \left(\frac{\partial R}{\partial y} \cos(\mathbf{n}, x) - \frac{\partial R}{\partial x} \cos(\mathbf{n}, y) \right) dS = \int_\Gamma R(x, y, z)\, dz \quad (5)$$

are proved in a similar way.

From (3), (4), (5) there follows Stokes' formula (1).

We have proved Stokes' theorem for an oriented surface element projectable simultaneously on each of the three coordinate planes. There is another important simple case when Stokes' theorem holds to which the above argument is inapplicable. Namely, we mean the case when σ^* is a part of a plane parallel to one of the coordinate axes. The validity of Stokes' theorem for such σ^* can be verified with the aid of direct computations similar to (3). However, it is also possible to use the following argument. The integrals entering into Stokes' formula are invariant with respect to the transformations of the rectangular coordinates preserving their orientation. A transformation of this type can always be chosen so that σ^* can be projected in a one-to-one manner on each of the coordinate planes of the new coordinate system (for instance, let us bring our plane into coincidence with the plane passing through the points (1, 0, 0), (0, 1, 0), (0, 0, 1) which proves the desired assertion.

Stokes' formula continues to hold for any oriented surface S^* (with a piecewise smooth edge Γ) which can be decomposed with the aid of piecewise smooth curves into a finite number of smooth pieces projectable on each of the three coordinate planes.

Indeed, let $S^* = \sigma_1^* + \ldots + \sigma_N^*$ be such a decomposition and let $\Gamma_1, \ldots, \Gamma_N$ be the coherently oriented contours bounding $\sigma_1^*, \ldots, \sigma_N^*$, respectively. Then, according

to what was proved above,

$$\iint_S (\operatorname{curl} \mathbf{a}\, d\sigma) = \sum_{j=1}^{N} \iint_{\sigma_j} (\operatorname{curl} \mathbf{a}\, d\sigma) = \sum_{j=1}^{N} \int_{\Gamma_j} (\mathbf{a}\, d\mathbf{l}) = \int_{\Gamma} (\mathbf{a}\, d\mathbf{l}),$$

since the parts of the integrals $\int_{\Gamma_j}$ $(j = 1, \ldots, N)$, corresponding to the arcs of Γ_j not belonging to Γ mutually cancel out because each of these arcs is traversed twice in the opposite directions.

An oriented surface which can be split into a finite number of (plane) triangles is called a *polyhedral* surface and is an example of a simple surface to which Stokes' formula is applicable.

We shall also note the following. Let σ_ε^* denote a circular plane area with centre at a point $A = (x, y, z)$ and radius ε oriented with the aid of its definitely chosen unit normal vector $\mathbf{n}$; let γ_ε be the oriented contour bounding σ_ε^*. According to Stokes' formula, we have

$$\int_{\gamma_\varepsilon} (\mathbf{a}\, d\mathbf{l}) = \int_{\sigma_\varepsilon} (\mathbf{n} \operatorname{curl} \mathbf{a})\, d\sigma_\varepsilon = \int_{\sigma_\varepsilon} \operatorname{curl}_{\mathbf{n}} \mathbf{a}\, d\sigma_\varepsilon = |\sigma_\varepsilon| \cdot \operatorname{curl}_{\mathbf{n}} \mathbf{a}_1,$$

where $\operatorname{curl}_{\mathbf{n}}\mathbf{a}$ is a scalar function equal to the projection of curl $\mathbf{a}$ on $\mathbf{n}$ and $\operatorname{curl}_{\mathbf{n}}\mathbf{a}_1$ is the value of this function assumed at some point belonging to σ_ε. Hence it follows that the value of the function $\operatorname{curl}_{\mathbf{n}}\mathbf{a}$ at the point A is expressed by the formula

$$\operatorname{curl}_{\mathbf{n}}\mathbf{a} = \lim_{\varepsilon \to 0} \frac{1}{|\sigma_\varepsilon|} \int_{\gamma_\varepsilon} (\mathbf{a}\, d\mathbf{l}), \tag{6}$$

where in the passage to the limit as $\varepsilon \to 0$ it is supposed that the vector $\mathbf{n}$ remains invariable. The right-hand side of (6) is a number which is the same in any right-handed (left-handed) coordinate system. However, if a right-handed system is replaced by a left-handed one while $\mathbf{n}$ is invariable, then the direction in which the contour γ_ε of σ_ε is described changes to the opposite and therefore the sign of the right-hand member of (6) also changes. This again shows, in another way, that curl $\mathbf{a}$ is invariant with respect

to the transformations preserving orientation of rectangular coordinates.

Remark. We should like to give a number of formulas involving the Hamiltonian operator ∇ which turn out to be useful in vector analysis.

We have

$$\operatorname{div} \mathbf{a} = (\nabla, \mathbf{a}), \quad \operatorname{curl} \mathbf{a} = \nabla \times \mathbf{a}, \quad \operatorname{grad} f = \nabla f,$$

where f is a scalar function and $\mathbf{a}$ is a vector.

(1) $$\operatorname{curl} [\operatorname{grad} f] \equiv \nabla \times \nabla f = 0,$$

since the symbolic vectors ∇ and ∇f differ only in a scalar factor. This fact is directly proved in Sec. 3.4, Theorem 2.

(2) $$\operatorname{div} \operatorname{curl} \mathbf{a} \equiv (\nabla, \nabla \times \mathbf{a}) = 0,$$

since the vector ∇ is orthogonal to the vector $\nabla \times \mathbf{a}$.

(3) $$\operatorname{div} (f\mathbf{a}) \equiv (\nabla, f\mathbf{a}) = (\nabla f, \mathbf{a}) + f(\nabla, \mathbf{a}).$$

Indeed,

$$(\nabla, f\mathbf{a}) = \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$

$$= f'_x P + f'_y Q + f'_z R + f(P'_x + Q'_y + R'_z) = (\nabla f, \mathbf{a}) + f(\nabla, \mathbf{a}).$$

(4) It is easily verified that $\operatorname{grad} fg = f \operatorname{grad} g + g \operatorname{grad} f$ or in the symbolic form $\nabla fg = f\nabla g + g\nabla f$. Thus, the operator ∇ acts on the product of two functions as an ordinary differentiation operator.

(5)

$$\operatorname{curl} (f\mathbf{a}) = f \operatorname{curl} \mathbf{a} + \operatorname{grad} f \times \mathbf{a} \text{ or } \nabla \times f\mathbf{a} = f(\nabla \times \mathbf{a}) + \nabla f \times \mathbf{a}.$$

(6) $$\operatorname{div} (\mathbf{a} \times \mathbf{b}) = (\mathbf{b}, \operatorname{curl} \mathbf{a}) - (\mathbf{a}, \operatorname{curl} \mathbf{b})$$

or

$$(\nabla, \mathbf{a} \times \mathbf{b}) = (\mathbf{b}, \nabla \times \mathbf{a}) - (\mathbf{a}, \nabla \times \mathbf{b}).$$

(7) $$\operatorname{div} \operatorname{grad} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \equiv \Delta f,$$

where Δ is called *Laplace's operator*. It is obvious that

$$(\nabla, \nabla f) = \Delta f.$$

CHAPTER 4

FOURIER SERIES. FOURIER INTEGRAL

Sec. 4.1. Trigonometric Series

A function $f(x)$ is said to be periodic (with period a) if it is defined throughout the real axis and satisfies the condition

$$f(x + a) = f(x)$$

for every x.

For instance, the trigonometric functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \ldots \tag{1}$$

are periodic with period 2π.

In fact, the functions $\cos kx$ and $\sin kx$ have period $2\pi/k$ for every natural k. Thus, $2\pi/k < 2\pi$ for $k > 1$. And the constant $y = 1$ has an arbitrarily small period. But all the functions of sequence (1) have a period 2π.

The periodic function

$$s = f(t)$$

represents periodic motion (oscillation) of a point having the coordinate s (on the x-axis) at an instant t.

The function (with period $2l$)

$$s = A \cos\left(\frac{k\pi}{l} t + \omega\right), \tag{2}$$

where $A > 0$, $l > 0$, and ω are constants, and k is natural, defines *harmonic oscillation* of a point with *amplitude* A, *phase* ω, and *frequency* k.

Function (2) has a period $2l/k$, i.e. one complete oscillation occurs during a time interval $2l/k$. And the number of oscillations per unit time is equal to $k/2l$. Just the number $k/2l$ should be termed the frequency of oscillation, but this term is usually given to the number k.

Note that the function

$$a_k \cos \frac{k\pi}{l} t + b_k \sin \frac{k\pi}{l} t \quad (\sqrt{a_k^2 + b_k^2} > 0),$$

where k is a natural number, defines harmonic oscillation since

$$a_k \cos \frac{k\pi}{l} t + b_k \sin \frac{k\pi}{l} t$$

$$= \sqrt{a_k^2 + b_k^2} \left(\frac{a_k}{\sqrt{a_k^2 + b_k^2}} \cos \frac{k\pi}{l} t + \frac{b_k}{\sqrt{a_k^2 + b_k^2}} \sin \frac{k\pi}{l} t \right)$$

$$= A_k \cos \left(\frac{k\pi}{l} t + \omega_k \right),$$

where

$$A_k = \sqrt{a_k^2 + b_k^2},$$

and ω_k is determined identically from the relationships

$$0 \leqslant \omega_k < 2\pi, \quad a_k / \sqrt{a_k^2 + b_k^2} = \cos \omega_k,$$

$$b_k / \sqrt{a_k^2 + b_k^2} = \sin \omega_k.$$

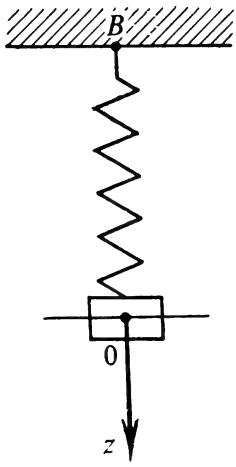

Fig. 105.

Harmonic oscillations can be exemplified by the oscillations of a spring pendulum shown diagrammatically in Fig. 105. Let the spring suspended at the point B have a load of mass m fixed at its lower end. The coordinate of the centre of gravity of the suspended load at the instant $t = 0$ is equal to $z = 0$. At the instant $t = 0$ a pulse $z' = \mu$ is imparted to the load in the direction of the z-axis. As a result, the load will start oscillating. Its deviation from the point of equilibrium will be denoted by $z = z(t)$. Since the force acting upon the load is equal (in the first approximation) to $mz'' = -kz$ (by Newton's law), then

$$z'' + \nu^2 z = 0 \quad \left(\nu^2 = \frac{k}{m} \right).$$

The general solution of this differential equation has the form

$$z = C_1 \cos \nu t + C_2 \sin \nu t,$$

where C_1, C_2 are arbitrary constants. Since

$$z(0)=0,\ z'(0) = \mu,$$

then

$$z=\frac{\mu}{\nu} \sin \nu t = A \cos\left(\nu t - \frac{\pi}{2}\right), \quad A=\frac{\mu}{\nu},$$

and we have obtained that the centre of gravity of the load performs harmonic oscillation.

Periodic motion (oscillation) is dealt with in various fields of knowledge: in the theory of elasticity, acoustics, radio engineering, electrical engineering, and everywhere harmonic oscillation is the simplest periodic motion.

The finite sum of the harmonic oscillations with the given period $2l$ represents a composite oscillation

$$S_n(t)=\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k \cos \frac{k\pi}{l} t + b_k \sin \frac{k\pi}{l} t\right). \qquad (3)$$

The zero term is written here in the form $a_0/2$. We shall see that this will prove convenient.

Finally, a more complex periodic oscillation (motion) can be obtained as the sum of the convergent (for all t) series

$$f(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k \cos \frac{k\pi}{l} t + b_k \sin \frac{k\pi}{l} t\right), \qquad (4)$$

called a *trigonometric series.*

The numbers a_k and b_k are called the *coefficients of trigonometric series* (4) and its individual terms

$$a_k \cos \frac{k\pi}{l} t + b_k \sin \frac{k\pi}{l} t$$

are the *terms of series* (4) or its *harmonics* (corresponding to the frequency k).

Example 1. Figures 106 to 109 represent the graphs of the first four partial sums of the series

$$\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} = \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots$$

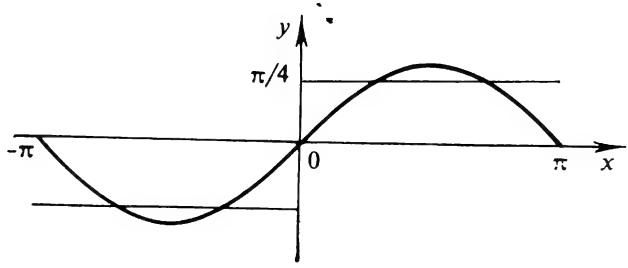

Fig. 106.

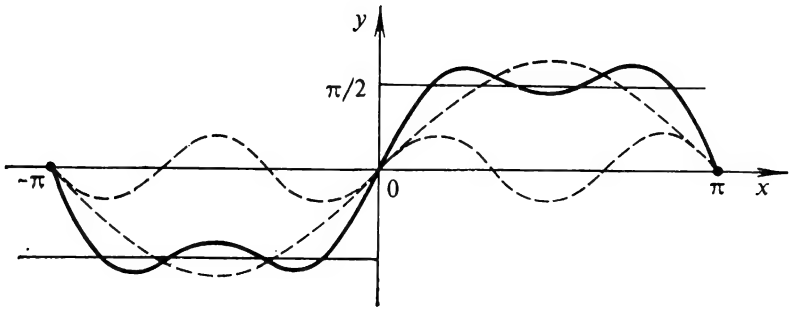

Fig. 107.

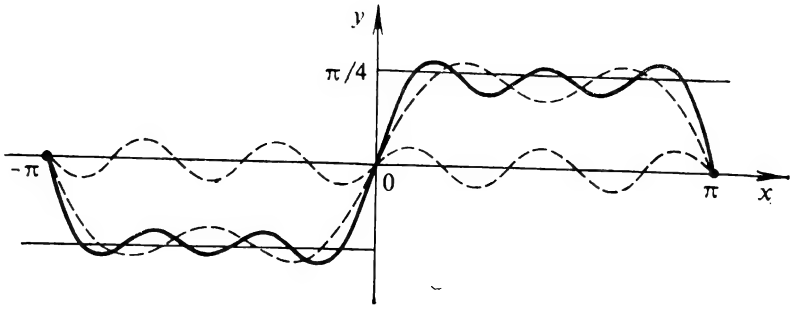

Fig. 108.

and the graph of the function

$$\psi(x)=\begin{cases} \pi/4, & 0<x<\pi, \\ 0, & x=0,\ x=\pi, \\ -\pi/4, & -\pi<x<0. \end{cases}$$

In Figure 106 (alongside with the graph of $\psi(x)$) the function $S_1(x) = \sin x$ is represented. In Fig. 107 the dashed

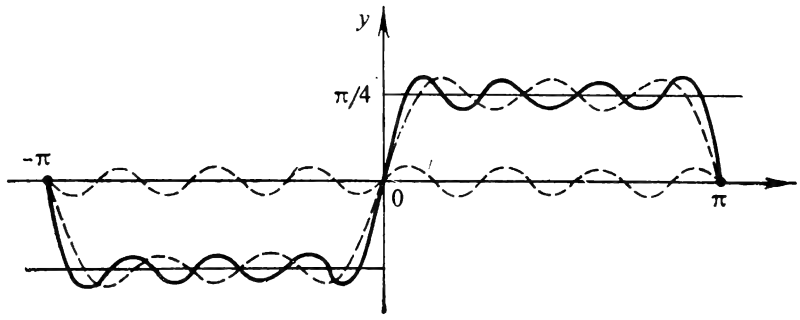

Fig. 109.

line represents the graphs of $S_1(x)$ and $\frac{\sin 3x}{3}$ and the continuous line the graph of

$$S_2(x) = S_1(x) + \frac{\sin 3x}{3}.$$

In Figure 108 the dashed line depicts the graphs of $S_2(x)$ and $\frac{\sin 5x}{5}$ and the continuous line that of

$$S_3(x) = S_2(x) + \frac{\sin 5x}{5}$$

and so on. As is already seen from Fig. 109, we have to assume

$$\lim_{n\to\infty} S_n(x) = \psi(x) \quad (-\pi < x \leqslant \pi). \tag{5}$$

This is just the case. As regards this, see below Sec. 4.4. The functions $S_n(x)$ have period 2π for any n:

$$S_n(x + 2\pi) = S_n(x).$$

Let us extend the function $\psi(x)$ periodically to the entire real axis with period 2π. Then it will have the graph as in

Fig. 110. Since equality (5) is fulfilled for all $x \in (-\pi, \pi)$ and the functions $S_n(x)$ and $\psi(x)$ have period 2π, we write

$$\lim_{n\to\infty} S_n(x) = \psi(x) \quad (-\infty < x < \infty).$$

Example 2. Figure 111 represents three periodic functions with period 2π:

$$S_2(x) = \sin x - \frac{\sin 2x}{2} \quad \text{(the continuous line)},$$

$$S_3(x) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \quad \text{(the dashed line)},$$

$$S_4(x) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} \quad \text{(the dotted line)}.$$

For large values of n the graph of the sum $S_n(x) = \sum_{k=1}^{n} (-1)^k \frac{\sin kx}{k}$ is shown schematically in Fig. 112 which suggests the idea that the limit function

$$S(x) = \lim_{n\to\infty} S_n(x)$$

is a periodic function (with period 2π) defined by the equalities

$$S(x) = \begin{cases} x, & -\pi < x < \pi, \\ 0 & x = \pi. \end{cases}$$

As we shall see below, this is just the case (see Sec. 4.4).

Sec. 4.2. Convergence of Trigonometric Series

Let there be given a trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{l} x + b_k \sin \frac{k\pi}{l} x \right). \tag{1}$$

To find out whether it is convergent, it is natural to consider the numerical series

$$\frac{|a_0|}{2} + \sum_{k=1}^{\infty} (|a_k| + |b_k|) \tag{2}$$

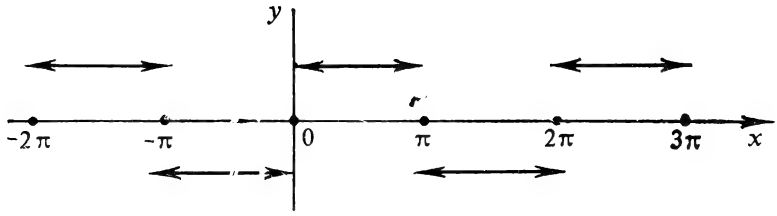

Fig. 110.

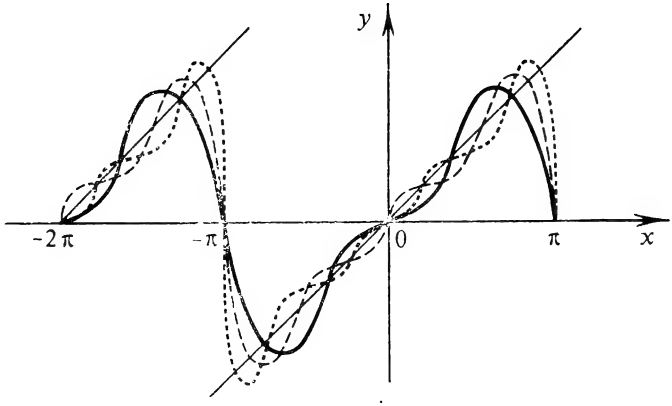

Fig. 111.

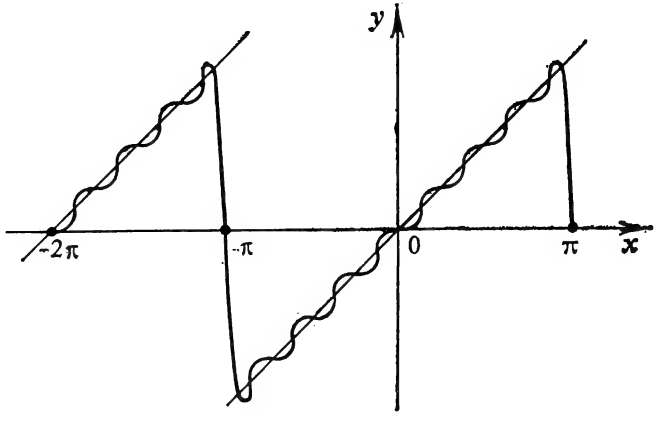

Fig. 112.

majorizing, as we say, series (1). Its terms exceed the absolute values of the corresponding terms of series (1):

$$\left|a_k \cos \frac{k\pi}{l} x\right| \leqslant \left|a_k\right|, \quad \left|b_k \sin \frac{k\pi}{l} x\right| \leqslant \left|b_k\right|.$$

Hence, it follows that if series (2) converges, then series (1) also converges for all x's, absolutely and uniformly (see our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 9.8, Theorem 1). But series (1) can converge without series (2) being convergent. As a matter of fact, as k varies, its terms for every x change sign (oscillate) an infinite number of times, and it may turn out to be convergent as a result of the positive terms being compensated by negative ones. In the general theory of series there are tests for convergence of such series, namely, Dirichlet's and Abel's* tests (see Sec. 9.9, Theorems 3 and 4, of the same book) which are very suitable for investigating trigonometric series.

In any case, if it is established that series (1) converges uniformly, then the fact that its terms are continuous functions having period $2l$ implies that its sum

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} + \left(a_k \cos \frac{k\pi}{l} x + b_k \sin \frac{k\pi}{l} x\right) \qquad (3)$$

is also a continuous function with period $2l$ (see Sec. 9.8, Theorem 2 and Sec. 9.9, Theorem 2, of the same book) and series (3) can be integrated termwise.

Series (3) can be formally differentiated with respect to x:

$$\sum_{k=1}^{\infty} \frac{k\pi}{l} \left(-a_k \sin \frac{k\pi}{l} x + b_k \cos \frac{k\pi}{l} x\right) \qquad (4)$$

and its majorizing series formed

$$\sum_{k=1}^{\infty} \frac{k\pi}{l} (|a_k| + |b_k|). \qquad (5)$$

Again, if series (5) converges, then series (4) converges uniformly. Moreover, on the basis of the known theorem from

* Abel, Niels Henrik (1802-1829). Norwegian mathematician. Dirichlet, Peter Gustav Lejeune (1805-1859). German mathematician.

the theory of uniformly convergent series, the sum of series (4) is then the derivative of the sum of series (3), i.e.

$$f'(x)=\sum_{k=1}^{\infty}\frac{k\pi}{l}\left(-a_k\sin\frac{k\pi}{l}x+b_k\cos\frac{k\pi}{l}x\right).$$

In general, if the series

$$\sum_{k=1}^{\infty}\left(\frac{k\pi}{l}\right)^s(|a_k|+|b_k|)<\infty$$

is convergent for some natural s, then series (3) can be differentiated termwise s times.

However, one should remember that it may be legitimate to differentiate series (3) one more time (that is, $s+1$ times).

Example 1. Find out how many times the series

$$\sum_{k=1}^{\infty}q^k\cos kx\quad(0<q<1)$$

can be differentiated termwise.

Let us differentiate formally the given series s times:

$$\pm\sum_{k=1}^{\infty}k^sq^k\begin{Bmatrix}\cos kx\\ \sin kx\end{Bmatrix}.$$

The majorizing series $\sum_{k=1}^{\infty}k^sq^k$ $(0<q<1)$ is convergent for any natural s which can be verified by D'Alembert's test. Therefore the initial series can be differentiated as many times as desired.

Problem 1. How many times can the following series be *a fortiori* differentiated termwise:

(a) $\sum_{k=1}^{\infty}\frac{\sin kx}{k^4}$, (b) $\sum_{k=1}^{\infty}\frac{\cos kx}{k^3}$,

c) $\sum_{k=1}^{\infty}q^k(a_k\cos kx+b_k\sin kx)$ $(0<q<1,$ $|a_k|,\ |b_k|<M).$

How many continuous derivatives have automatically the sums of these series? (See also Example 1 from Sec. 9.9 of the above mentioned book.)

Sec. 4.3. Fourier Series

Let there be given a function $f(t)$ with period $2l$ and let it be known that the given function can be expanded in a trigonometric series

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{l} t + b_k \sin \frac{k\pi}{l} t \right), \qquad (1)$$

i.e. it is already the sum of a trigonometric series (of form (1)) for all t (or, possibly, for all t with the exception of individual values of t). The question is how to determine the coefficients a_k, b_k, given the function $f(t)$. This question was answered by mathematicians and physicists at the beginning of the last century. A valuable contribution to the solution of this problem was made by J. Fourier*. He showed that the coefficients a_k, b_k of a trigonometric series representing a periodic function $f(t)$ having a period of $2l$ are computed by the formulas

$$\left.\begin{aligned} a_k &= \frac{1}{l} \int_{-l}^{l} \cos \frac{k\pi}{l} t\, f(t)\, dt \quad (k = 0, 1, 2, \ldots), \\ b_k &= \frac{1}{l} \int_{-l}^{l} \sin \frac{k\pi}{l} t\, f(t)\, dt \quad (k = 1, 2, \ldots). \end{aligned}\right\} \qquad (2)$$

The numbers a_k and b_k, computed by these formulas, are called the *Fourier coefficients* of the function $f(t)$ and trigonometric series (1) into which instead of a_k and b_k the corresponding Fourier coefficients are substituted is called the *Fourier series* of the function $f(t)$.

In some cases (for more narrow classes of functions) formulas (2) were already known to Euler, therefore they are also termed the Euler-Fourier formulas.

---

* Fourier, Jean Baptiste Joseph (1768-1830). French mathematician.

Formulas (2) will be derived in Sec. 4.6 in the assumption that it is already known that a periodic function $f(t)$ having period $2l$ can be expanded in a trigonometric series uniformly convergent to this function.

It should be mentioned here that for long physicists held that any compound periodic motion of a point (complex oscillation), though it were mechanical vibrations of a point of a sounding string, or electromagnetic oscillation, or vibrations caused by sound propagation, is decomposed into harmonic oscillations (vibrations), i.e. a compound periodic motion should be understood as a sum (finite or infinite) of simple harmonic vibrations of the same period. Singling out of a compound periodic motion a component harmonic oscillation corresponding to a given frequency k is of great practical importance. Physicists obtain such a separation of harmonic oscillation from a real motion with the aid of special devices called resonators. Given a motion defined by a periodic function $s = f(t)$, a mathematician obtains such a separation by means of computations. He simply computes the Fourier coefficients a_k, b_k of the given function, and then the corresponding kth harmonic will have the form

$$a_k \cos \frac{k\pi}{l} t + b_k \sin \frac{k\pi}{l} t.$$

Note that if a function $f(t)$ has period a and is integrable on the closed interval $[0, a]$ or, as we say, on the period, then the following equality holds true for this function

$$\int_0^a f(x)\,dx = \int_\lambda^{a+\lambda} f(x)\,dx \qquad (3)$$

for any real number λ (see our book *Higher Mathematics, Differential and Integral Calculus*, Sec. 6.4, Example 8).

Property (3), in particular, shows that the Fourier coefficients of the periodic function $f(t)$ with period $2l$ can be written in the form

$$a_k = \frac{1}{l} \int_\lambda^{\lambda+2l} f(t) \cos \frac{k\pi}{l} t\,dt \quad (k = 0,\ 1,\ 2,\ \ldots),$$

$$b_k = \frac{1}{l} \int_\lambda^{\lambda+2l} f(t) \sin \frac{k\pi}{l} t\,dt \quad (k = 1,\ 2,\ \ldots),$$

where λ is an arbitrary real number, since the functions $\cos \frac{k\pi}{l} t$ and $\sin \frac{k\pi}{l} t$ have period $2l$, and the product of functions with period $2l$ is, in its turn, a function having period $2l$.

One more note: if the function f is even on the interval $[-a, a]$, then (see Sec. 6.4, Example 6, of the above mentioned book)

$$\int_{-a}^{a} f(x)\, dx = 2 \int_{0}^{a} f(x)\, dx.$$

And if the function f is odd on the interval $[-a, a]$, then (see Sec. 6.4, Example 7, of the same book)

$$\int_{-a}^{a} f(x)\, dx = 0.$$

The function $\cos \frac{k\pi}{l} t$ is even and $\sin \frac{k\pi}{l} t$ is odd. Besides, the product of two even or two odd functions is an **even**

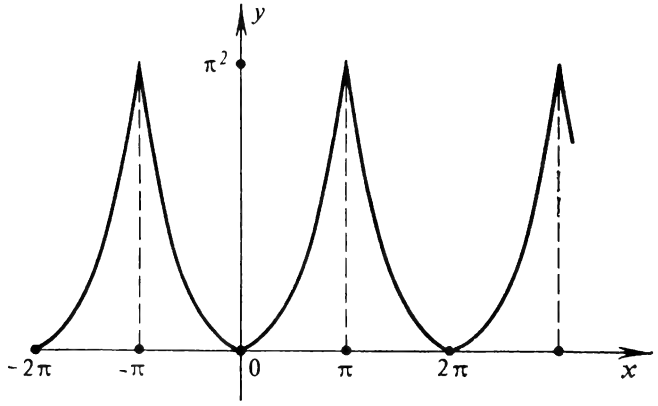

Fig. 113.

function, and the product of an even function by an odd one is an odd function. Therefore, for an even function $f(t)$ having period $2l$

$$a_k = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{k\pi}{l} x\, dx \qquad (k = 0, 1, 2, \ldots),$$

$$b_k = 0 \qquad (k = 1, 2, \ldots),$$

and for an odd one

$$a_k = 0 \qquad (k = 0,\ 1,\ 2,\ \ldots),$$

$$b_k = \frac{2}{l}\int_0^l f(x)\sin\frac{k\pi}{l}x\,dx \qquad (k = 1,\ 2,\ \ldots).$$

Example 1. Expand the function $f(x) = x^2$ (with period 2π $(-\pi \leqslant x \leqslant \pi)$ in a Fourier series.

The given function is an even one. Then its Fourier series consists only of cosines ($b_k = 0$). Let us compute the coefficients a_k:

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^\pi x^2\,dx = \frac{\pi^2}{3}$$

and

$$a_k = \frac{2}{\pi}\int_0^\pi x^2\cos kx\,dx = \frac{2}{\pi}\,x^2\frac{\sin kx}{k}\Big|_0^\pi - \frac{4}{k\pi}\int_0^\pi x\sin kx\,dx$$

$$= \frac{4x}{\pi k}\,\frac{\cos kx}{k}\Big|_0^\pi - \frac{4}{\pi k^2}\int_0^\pi \cos kx\,dx = (-1)^k\,\frac{4}{k^2} \qquad (k > 0).$$

Thus,

$$x^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}(-1)^k\,\frac{\cos kx}{k^2} \qquad (-\pi \leqslant x \leqslant \pi).$$

The graph of the sum of this series is represented in Fig. 113.

Sec. 4.4. The Tests for Convergence of Fourier Series

To simplify the notation, we shall consider functions with period 2π. The functions with period $2l$ are dealt with in a similar way.

As we know, the Fourier series of a function $f(x)$ with period 2π is defined as the trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx), \tag{1}$$

whose coefficients are computed by the formulas

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx \qquad (k = 0,\ 1,\ 2,\ \ldots),$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx \qquad (k = 1,\ 2,\ \ldots). \tag{2}$$

We see that for the Fourier series of the function f with period 2π to have sense, it is necessary that integrals (2) be meaningful in any case.

In our considerations integrals (2) will always have sense, since we shall speak of continuous or piecewise continuous functions bounded on their periods.

Let us now ask the following question: what conditions must be satisfied by the function $f(x)$ so that its Fourier series converges to it? Many mathematicians tried to give an

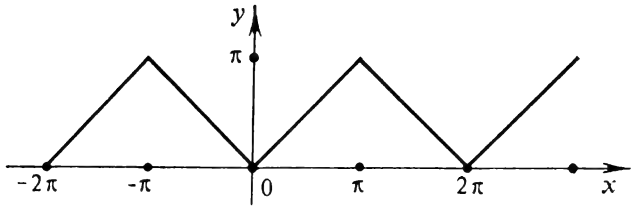

Fig. 114.

exhaustive answer to this question. Here we shall confine ourselves to formulating several tests of practical importance for convergence of Fourier series without proving them.

If a function $f(x)$ with period 2π is continuous throughout the real axis and has a piecewise continuous derivative on its period, then its Fourier series uniformly converges to it:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Example 1. The function $\psi(x)$ with period 2π, even and defined on the interval $[0, \pi]$ by the equality

$$\psi(x) = x \qquad (0 \leqslant x \leqslant \pi),$$

satisfies, obviously, the formulated test (see the graph of this function in Fig. 114). Its Fourier coefficients $b_k = 0$ and the coefficients

$$a_k = \frac{2}{\pi}\int_0^{\pi} x\cos kx\,dx$$

$$= \frac{2}{\pi}\,x\,\frac{\sin kx}{k}\Big|_0^{\pi} - \frac{2}{\pi k}\int_0^{\pi}\sin kx\,dx = \frac{2}{\pi k^2}\cos kx\Big|_0^{\pi}$$

$$= \frac{2}{\pi k^2}(\cos k\pi - 1), \quad \frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}.$$

Thus,

$$a_0 = \pi, \quad a_{2k} = 0, \quad a_{2k-1} = \frac{-4}{\pi(2k-1)^2} \quad (k = 1, 2, \ldots).$$

But then, according to the indicated test,

$$\psi(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2} \qquad (-\infty < x < \infty);$$

the convergence of the series being uniform.

The fact that this series is uniformly convergent also follows from the general theory of series (according to Weierstrass'* M-test). But it is not trivial that the series converges just to the function $\psi(x)$.

Let us also note that the given periodic function $\psi(x)$ coincides with the function $y = x$ only on the closed interval $[0, \pi]$, whereas outside this interval the two functions are different.

Let us also formulate the test for convergence of a Fourier series called the *Dirichlet test.*

A periodic function $f(x)$ with period 2π is said to satisfy the Dirichlet condition if on the interval $[0, 2\pi]$ we can indicate a finite number of points $0 = x_0 < x_1 < x_2 < \ldots < x_N = 2\pi$ such that on the intervals (x_j, x_{j+1}) the function is bounded and monotone (not decreasing or not

* Weierstrass, Karl Theodor Wilhelm (1815-1897). Outstanding German mathematician.

increasing), and at every point x_j of discontinuity of f

$$f(x_j) = \frac{1}{2}(f(x_j + 0) + f(x_j - 0)),$$

i.e. the value of f at x_j is an arithmetic mean of the right-hand and left-hand limits of f at x_j.

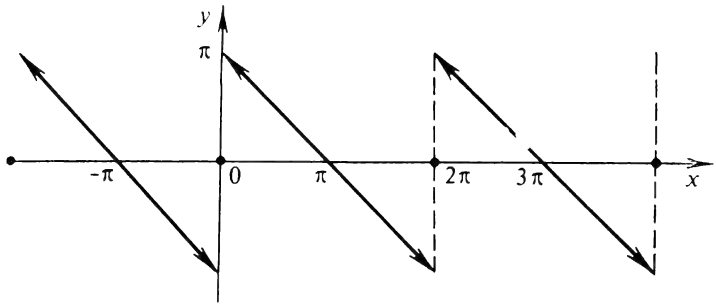

Fig. 115.

If the function $f(x)$ having period 2π satisfies the Dirichlet condition, then its Fourier series converges to it for any x:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \qquad (-\infty < x < \infty).$$

Example 2. As it is seen from its graph (Fig. 115), the functions $\varphi(x)$ with period 2π defined by the equality

$$\varphi(x) = \begin{cases} \pi - x, & 0 < x < 2\pi, \\ 0 & x = 0, \end{cases}$$

satisfies the Dirichlet condition. In fact, the points $0 = x_0 < x_1 = 2\pi$ possess the following property: the function $\varphi(x)$ is decreasing and bounded on the interval (x_0, x_1) and

$$\varphi(0) = \frac{\varphi(0+0) + \varphi(0-0)}{2} = \frac{\pi + (-\pi)}{2} = 0,$$

$$\varphi(2\pi) = \frac{\varphi(2\pi+0) + \varphi(2\pi-0)}{2} = \frac{\pi + (-\pi)}{2} = 0.$$

The function $\varphi(x)$ is odd, therefore its Fourier series consists only of sines; consequently, the Fourier coefficients for it

$$a_k = 0 \qquad (k = 0,\ 1,\ 2,\ \ldots);$$

$$b_k = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin kx\,dx$$

$$= -\frac{2}{\pi}(\pi - x)\frac{\cos kx}{k}\Big|_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\cos kx}{k}dx$$

$$= \frac{2}{k} - \frac{2}{\pi k}\cdot\frac{\sin kx}{k}\Big|_0^{\pi} = \frac{2}{k} \qquad (k = 1,\ 2,\ \ldots).$$

Thus,

$$\varphi(x) = 2\sum_{k=1}^{\infty}\frac{\sin kx}{k} \qquad (-\infty < x < \infty).$$

Problem 1. Expand in a Fourier series the function with

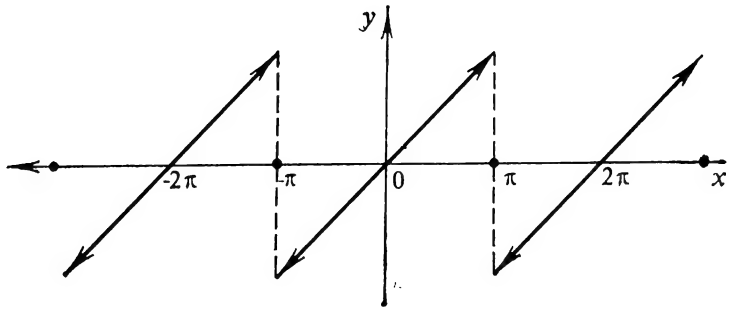

Fig. 116.

period 2π defined on $[-\pi, \pi]$ by the equality (Fig. 116)

$$f(x) = \begin{cases} x, & -\pi < x < \pi, \\ 0, & x = \pm\pi. \end{cases}$$

Answer. $f(x) = 2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin kx}{k}$.

Sec. 4.5. Orthogonal Properties of Trigonometric Functions

Consider the sequence of trigonometric functions

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots . \tag{1}$$

The following important and readily verified formulas hold true for these functions:

$$\left.\begin{aligned}
&\int_{-\pi}^{\pi} \cos kx \cos lx\, dx = \begin{cases} 0, & k \neq l, \\ \pi, & k = l, \end{cases} \\
&\int_{-\pi}^{\pi} \sin kx \sin lx\, dx = \begin{cases} 0, & k \neq l, \\ \pi, & k = l, \end{cases} \\
&\int_{-\pi}^{\pi} \cos kx \cdot \sin lx\, dx = 0 \quad (k,\ l = 1,\ 2,\ 3,\ \ldots), \\
&\int_{-\pi}^{\pi} \cos kx\, dx = 0 \quad (k = 1,\ 2,\ \ldots), \\
&\int_{-\pi}^{\pi} \sin kx\, dx = 0 \quad (k = 1,\ 2,\ \ldots), \\
&\int_{-\pi}^{\pi} 1\, dx = 2\pi.
\end{aligned}\right\} \tag{2}$$

It follows, in particular, from (2) that the integral over the interval $[-\pi, \pi]$ of the product of any two different functions of sequence (1) is equal to zero.

This property is formulated in the following way: *The functions of sequence* (1) *are orthogonal on the interval* $[-\pi, \pi]$.

Besides, it follows from formulas (2):

$$\left.\begin{aligned}
&\int_{-\pi}^{\pi} \cos^2 kx\, dx = \pi, \\
&\qquad\qquad k = 1,\ 2,\ \ldots, \\
&\int_{-\pi}^{\pi} \sin^2 kx\, dx = \pi, \\
&\int_{-\pi}^{\pi} 1^2\, dx = 2\pi.
\end{aligned}\right\} \tag{3}$$

Problem 1. Derive the formulas analogous to formulas (2), (3) for the trigonometric functions

$$1,\ \cos\frac{\pi}{l}x,\ \sin\frac{\pi}{l}x,\ \cos\frac{2\pi}{l}x,\ \sin\frac{2\pi}{l}x,\ \ldots$$

Hint. The required formulas can be obtained by direct computation. But they can also be derived by changing the variable $x=\frac{\pi u}{l}$ in integrals (2) and (3).

Sec. 4.6. Fourier Coefficients

Suppose that a function $f(x)$ with period 2π is expanded in the trigonometric series

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx), \qquad (1)$$

and it turns out that this series converges uniformly to the given function.

Each term of series (1) is a continuous function and, since series (1) is uniformly convergent (by the hypothesis), its sum $f(x)$ is a continuous (on the real axis) function (see our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 9.8, Theorem 2).

Let us multiply both sides of (1) by $\cos mx$, where m is a natural number. Since the function $\cos mx$ is continuous and bounded, the obtained series will also consist of continuous functions and once again converge uniformly, but this time to the continuous function $f(x)\cos mx$. And it is legitimate to integrate uniformly convergent series termwise on a finite interval. Let us integrate the obtained series termwise on the period, i.e. on the closed interval $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi}f(x)\cos mx\,dx=\int_{-\pi}^{\pi}\frac{a_0}{2}\cos mx\,dx$$

$$+\sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kx\cos mx\,dx+b_k\int_{-\pi}^{\pi}\sin kx\cos mx\,dx\right)$$

$$=a_m\cdot\pi\quad(m=1,\ 2,\ \ldots).$$

The second equality follows from formulas (2) given in the preceding section (the orthogonal property of trigonometric functions).

Analogously, we obtain (by virtue of the last three formulas (2) from the preceding section)

$$\int_{-\pi}^{\pi} f(x) \sin mx\, dx = b_m \pi \quad (k = 1, 2, \ldots)$$

and

$$\int_{-\pi}^{\pi} f(x)\, 1\, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} 1\, dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kx\, dx + b_k \int_{-\pi}^{\pi} \sin kx\, dx \right) = \frac{a_0}{2} 2\pi = a_0 \pi.$$

As it was already noted in Sec. 4.5 (2), the numbers a_m, b_m computed by formulas (2) are called the Fourier coefficients of the function f, and the trigonometric series (1) itself, where a_k and b_k are the Fourier coefficients of the function f is called the Fourier series of the function f.

Thus, we have proved that *if a function f is representable in the form of the sum of trigonometric series* (1), *uniformly convergent (for all x's!), then the numbers a_k, b_k are necessarily the Fourier coefficients of the function f.*

Remark 1. Thus, any uniformly convergent trigonometric series is a Fourier series of its sum.

Remark 2. We have considered here the function $f(x)$ with period 2π in order to avoid complicated notation. For period $2l$ the arguments are analogous.

Sec. 4.7. Estimating the Fourier Coefficients

Theorem 1. *Let a function $f(x)$ with period 2π have a continuous derivative $f^{(s)}(x)$ of order s satisfying throughout the real axis the inequality*

$$|f^{(s)}(x)| \leqslant M_s; \tag{1}$$

then the Fourier coefficients of the function f satisfy the inequality

$$|a_k| \leqslant \frac{2M_s}{k^s}, \quad |b_k| \leqslant \frac{2M_s}{k^s} \quad (k = 1, 2, \ldots). \tag{2}$$

Proof. Integrating by parts and taking into account that $f(-\pi) = f(\pi)$, we have

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt$$

$$= \frac{1}{\pi} \left[f(t) \frac{\sin kt}{k} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(t) \frac{\sin kt}{k} dt \right]$$

$$= -\frac{1}{\pi k} \int_{-\pi}^{\pi} f'(t) \sin kt \, dt. \qquad (3)$$

Therefore

$$|a_k| \leqslant \frac{1}{\pi k} \int_{-\pi}^{\pi} M_1 \cdot 1 \, dx = \frac{2M_1}{k}.$$

Integrating by parts the right-hand side of (3) in succession, taking into account that the derivatives $f', \ldots, f^{(s-1)}$ are continuous and attain the same values at the points $t = -\pi$ and $t = \pi$, and also estimation (1), we obtain the first estimation in (2).

The second estimation in (2) is obtained in a similar way.

Sec. 4.8. Space of Functions with Scalar Product

A function $f(x)$ is called piecewise continuous on the interval $[a, b]$ if it is continuous on this interval except, possibly, at a finite number of points, where it has discontinuities of the first kind (see also Sec. 7.4 of our book *Higher Mathematics. Differential and Integral Calculus*). Such functions can be added and multiplied by real numbers; as a result, we again obtain functions piecewise continuous on $[a, b]$.

A scalar product of two functions f and φ, piecewise continuous on $[a, b]$ $(a < b)$ is defined as the integral

$$(f, \varphi) = \int_a^b f(x) \varphi(x) \, dx. \qquad (1)$$

Obviously, for any functions f, φ, ψ piecewise continuous on $[a, b]$ the following properties are fulfilled:

(1) $(f, \varphi) = (\varphi, f)$.

(2) $(f, f) \geqslant 0$ and it follows from the equality $(f, f) = 0$ that $f(x) = 0$ on $[a, b]$ except possibly for a finite number of points x.

(3) $(\alpha f + \beta\varphi, \psi) = \alpha (f, \varphi) + \beta (\varphi, \psi)$,

where α, β are arbitrary real numbers.

The set of all piecewise continuous functions defined on an interval $[a, b]$ for which a scalar product is introduced according to formula (1) will be denoted $L_2' = L_2'(a, b)$ and called space L_2' or $L_2'(a, b)$.

Remark 1. In mathematics the space $L_2 = L_2(a, b)$ is defined as the totality of functions $f(x)$ integrable in the Lebesgue sense on $[a, b]$ together with their squares for which scalar product is introduced according to formula (1). The space L_2' in question is a part of L_2. The L_2' space possesses many properties of the L_2 space, but not all of them (see the remark of the next section).

From properties (1), (2), (3) there follows *Buniakowski's* inequality* which is of great importance (see our book *Higher Mathematics. Fundamentals of Linear Algebra and Analytical Geometry*, Sec. 62, (6))

$$|(f, \varphi)| \leqslant (f, f)^{1/2} (\varphi, \varphi)^{1/2},$$

which in terms of integrals is written as follows:

$$\left| \int_a^b f(x) \varphi(x) \, dx \right| \leqslant \sqrt{\int_b^a f^2(x) \, dx} \sqrt{\int_b^a \varphi^2(x) \, dx}.$$

The quantity

$$\| f \| = \left(\int_b^a f^2(x) \, dx \right)^{1/2} = (f, f)^{1/2}$$

is called the *norm* of the function f.

* Buniakowski, Victor Yakovlevich (1804-1889). Noted Russian mathematician, made contributions to number theory and probability. *Translator's note*: the names Schwarz, Cauchy and Cauchy-Schwarz inequality are also encountered in the literature.

The norm has the following properties:

(1) $$\| f \| \geqslant 0,$$

where an equality can be only for the zero function $f = 0$, i.e. for a function equal to zero except possibly for a finite number of points (see our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 6.2, Theorem 5),

(2) $$\| f + \varphi \| \leqslant \| f \| + \| \varphi \|$$

(3) $$\| \alpha f \| = | \alpha | \cdot \| f \|,$$

where α is a real number.

In terms of integrals the second property appears as follows:

$$\left(\int_a^b | f(x) + \varphi(x) |^2 \, dx \right)^{1/2} \leqslant \left(\int_a^b f^2(x) \, dx \right)^{1/2} + \left(\int_a^b \varphi^2(x) \, dx \right)^{1/2}$$

and is called *Minkowski's* inequality.*

A sequence of functions $\{f_n\}$, *belonging to* L_2', *is said to be convergent to the function* $f \in L_2'$ *in the sense of a mean square on* $[a, b]$ *(or according to the norm of* L_2'*) if*

$$\lim_{n \to \infty} \| f_n - f \| = \lim_{n \to \infty} \left(\int_a^b | f_n(x) - f(x) |^2 \, dx \right)^{1/2} = 0.$$

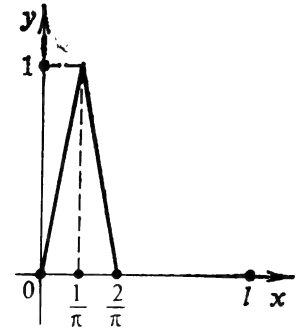

Fig. 117.

Note that if a sequence of functions $f_n(x)$ converges uniformly to the function $f(x)$ on the interval $[a, b]$, then for sufficiently large n the absolute value of the difference $f(x) - f_n(x)$ must be small for all $x \in [a, b]$.

And if $f_n(x)$ tends to $f(x)$ in the mean-square sense on the interval $[a, b]$, then the indicated difference may be not small for large n everywhere on $[a, b]$. At separate places of the interval $[a, b]$ this difference may be large, but what is

* Minkowski, Hermann (1864-1909). Number theorist, algebraist, analyst, and geometer. Born in Russia, lived in Switzerland and Germany.

of importance is that the integral of its square over the interval $[a, b]$ be small for large n.

Example 1. Let on $[0, 1]$ there be given a continuous piecewise linear function $f_n(x)$ $(n = 1, 2, \ldots)$ shown in Fig. 117, where

$$f_n(0) = f_n(2/n) = f_n(1) = 0,$$

$$f_n(1/n) = 1.$$

For any natural n

$$\max_{0 \leqslant x \leqslant 1} f_n(x) = 1,$$

and, consequently, this sequence of functions is not uniformly convergent to zero as $n \to \infty$.

Meanwhile,

$$\| f_n - 0 \| = \| f_n \| = \left(\int_0^1 f_n^2(x)\, dx \right)^{1/2}$$

$$= \left(\int_0^{1/n} (nx)^2\, dx + \int_{1/n}^{2/n} (2 - nx)^2\, dx \right)^{1/2}$$

$$= \left(2 \int_0^{1/n} (nx)^2\, dx \right)^{1/2} = \left(\frac{2}{3n} \right)^{1/2} \to 0, \quad n \to \infty,$$

i.e. the sequence of functions $\{f_n(x)\}$ tends to zero in the mean-square sense on $[0, 1]$.

Using the terms of the sequence $f_1, f_2, f_3, \ldots$, we can form a series

$$f_1 + f_2 + f_3 + \ldots \tag{2}$$

The sum of its first n terms

$$\sigma_n = f_1 + f_2 + \ldots + f_n$$

is a function belonging to L_2'. If it happens that in L_2' there exists a function f such that

$$\| f - \sigma_n \| \to 0, \quad n \to \infty,$$

then series (2) is said to be convergent to the function f in the mean-square sense and we write

$$f = f_1 + f_2 + f_3 + \ldots$$

Remark 2. It is possible to consider the space $L_2' = L_2'(a, b)$ of complex-valued functions $f(x) = f_1(x) + if_2(x)$, where f_1 and f_2 are real functions piecewise continuous on $[a, b]$. In this space the functions are multiplied by complex numbers and the scalar product of the functions $f(x) = f_1(x) + if_2(x)$ and $\varphi(x) = \varphi_1(x) + i\varphi_2(x)$ is determined in the following way:

$$(f, \varphi) = \int_a^b f(x)\,\overline{\varphi(x)}\,dx = \int_a^b [f_1(x) + if_2(x)]\,[\varphi_1(x) - i\varphi_2(x)]\,dx,$$

and the norm of f is determined as the quantity

$$\|f\| = (f, f)^{1/2} = \left(\int_a^b f \cdot \overline{f}\,dx\right)^{1/2} = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$$

$$= \left(\int_a^b [f_1^2(x) + f_2^2(x)]\,dx\right)^{1/2}.$$

Sec. 4.9. Orthogonal System of Functions

A function $\varphi \in L_2' = L_2'(a, b)$ is called *normal* if

$$\|\varphi\| = (\varphi, \varphi)^{1/2} = 1.$$

Two functions $\varphi, \psi \in L_2'$ are called *mutually orthogonal* if $(\varphi, \psi) = 0$.

A system of continuous (on the interval $[a, b]$) functions

$$\varphi_1, \varphi_2, \varphi_3, \ldots \tag{1}$$

(finite or infinite) is termed *orthogonal* if the functions have a positive norm and are pairwise orthogonal.

System (1) is called *orthogonal* and *normal* (*orthonormal*) or *orthonormalized* if

$$(\varphi_k, \varphi_l) = \delta_{kl} = \begin{cases} 0, & k \neq l, \\ 1, & k = l, \end{cases}$$

i.e. it is orthogonal and each entering into it function has a unit norm.

Any finite orthogonal system of functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ is linearly independent in L_2', that is from the fact that

$$\sum_{k=1}^{N} \alpha_k \varphi_k(x) = 0, \quad x \in [a, b],$$

where α_k are numbers, it follows that all $\alpha_k = 0$. Indeed, if we multiply (as scalars) both sides of this equality by φ_l $(l = 1, \ldots, N)$, then on the basis of the linear properties of a scalar product, we obtain

$$\left(\sum_{k=1}^{N} \alpha_k \varphi_k, \ \varphi_l\right) = \alpha_l (\varphi_l, \ \varphi_l) = 0,$$

and since $(\varphi_l, \varphi_l) > 0$, then $\alpha_l = 0$ $(l = 1, \ldots, N)$.

If $f \in L_2' = L_2'(a, b)$ is an arbitrary function, then the number

$$\frac{1}{\| \varphi_k \|^2} (f, \ \varphi_k) \quad (k = 1, \ 2, \ \ldots)$$

is called the Fourier coefficient of the function f with respect to the function φ_k of orthogonal system (1). The series

$$f \sim \sum_{k=1}^{\infty} \frac{1}{\| \varphi_k \|^2} (f, \ \varphi_k) \varphi_k, \tag{2}$$

generated by the function $f \in L_2'$ is called the Fourier series of the function f according to orthogonal system (1).

If system (1) is orthonormal, then $\| \varphi_k \| = 1$ $(k = 1, 2, \ldots)$ and the Fourier series of the function f is written in a simpler manner:

$$f \sim \sum_{k=1}^{\infty} (f, \ \varphi_k) \varphi_k. \tag{3}$$

In this case the Fourier coefficients are the numbers (f, φ_k). Below, we shall consider only orthonormalized systems (1). The passage from them to arbitrary orthogonal systems is a matter of techniques.

Theorem 1. *If system* (1) *is orthonormalized, then for any function* $f \in L_2'$ *the norm*

$$\left\| f - \sum_{k=1}^{N} \alpha_k \varphi_k \right\|$$

among all possible systems of numbers $\alpha_1, \ldots, \alpha_N$ *reaches its minimum for a unique system of numbers defined by the*

equalities

$$\alpha_k = (f,\ \varphi_k)\ (k = 1,\ \ldots,\ N),$$

i.e. for the Fourier coefficients of the function f.

Thus,

$$\min_{\alpha_k} \left\| f - \sum_{k=1}^{N} \alpha_k \varphi_k \right\| = \left\| f - \sum_{k=1}^{N} (f,\ \varphi_k)\, \varphi_k \right\|, \tag{4}$$

and

$$\left\| f - \sum_{k=1}^{N} (f,\ \varphi_k)\, \varphi_k \right\|^2 = (f,\ f) - \sum_{k=1}^{N} (f,\ \varphi_k)^2. \tag{5}$$

Proof. We have

$$\left\| f - \sum_{k=1}^{N} \alpha_k \varphi_k \right\|^2 = \left(f - \sum_{k=1}^{N} \alpha_k \varphi_k,\ f - \sum_{j=1}^{N} \alpha_j \varphi_j \right)$$

$$= (f,\ f) - 2 \sum_{k=1}^{N} \alpha_k (f,\ \varphi_k) + \sum_{k=}^{N} \alpha_k^2$$

$$= \sum_{k=1}^{N} [(f,\ \varphi_k)^2 - 2\alpha_k (f,\ \varphi_k) + \alpha_k^2] + (f,\ f) - \sum_{k=1}^{N} (f,\ \varphi_k)^2$$

$$= \sum_{k=1}^{N} [(f,\ \varphi_k) - \alpha_k]^2 + (f,\ f) - \sum_{k=1}^{N} (f,\ \varphi_k)^2$$

$$\geqslant (f,\ f) - \sum_{k=1}^{N} (f,\ \varphi_k)^2.$$

It is obvious here that the last relation in this chain turns into an equality only in one case, when $\alpha_k = (f,\ \varphi_k)$ for any k. We have thus proved relationships (4) and (5).

If we take into account that its left-hand side is a non-negative number, equality (5) yields the inequality

$$\sum_{k=1}^{N} (f,\ \varphi_k)^2 \leqslant (f,\ f),$$

which is true for any N. But then if system (1) consists of an infinite number of functions φ_k, the series made up of the squares of the Fourier coefficients of the function f is convergent and the following inequality holds true:

$$\sum_{k=1}^{\infty} (f,\ \varphi_k)^2 \leqslant (f,\ f). \tag{6}$$

It is known as *Bessel inequality.*

Of importance is the case when orthonormalized system (1) such that inequality (6) turns into an equality (the so-called *Parseval-Steklov* equality*)

$$\sum_{k=1}^{\infty} (f, \varphi_k)^2 = (f, f) \tag{7}$$

for all functions $f \in L_2'$.

To explain the meaning of the Parseval equality, let us take an arbitrary function $f \in L_2'$ and form its Fourier series

$$f \sim \sum_{k=1}^{\infty} (f, \varphi_k) \varphi_k.$$

The sum of the first n terms of this **series**

$$S_n(x) = \sum_{k=1}^{n} (f, \varphi_k) \varphi_k(x)$$

is called the nth *Fourier sum of the function* f *according to orthogonal system* (1).

According to formula (5), the deviation of $S_n(x)$ from $f(x)$ in the mean square (in the L_2' sense) is equal to

$$\| f - S_n \|^2 = (f, f) - \sum_{k=1}^{n} (f, \varphi_k)^2. \tag{8}$$

If for the function $f \in L_2'$ the Parseval equality is fulfilled, then

$$\| f - S_n \| \to 0, \quad n \to \infty, \tag{9}$$

and, conversely, from (9) there follows the validity of Parseval equality (7).

There exists the following terminology. Orthogonal system (1) is said to be complete in L_2' if the Fourier series of any function $f \in L_2'$ converges in the mean square to f, i.e. if property (9) holds for all $f \in L_2'$.

Thus, we have proved that *for orthogonal system* (1) *to be complete in* L_2' *it is necessary and sufficient that Parseval's relation* (7) *be fulfilled for any function* $f \in L_2'$.

---

* Parseval des Chenes, Marc Antoine (1755-1836). French mathematician.

Steklov, Vladimir Andreevich (1864-1926). Russian mathematician and physicist.

Note. We already noted in Remark 1 of the preceding section that $L_2 = L_2(a, b)$ denotes the space of functions $f(x)$ integrable in the Lebesgue sense on $[a, b]$ together with their squares and that $L_2' \subset L_2$.

Consider an orthonormalized on $[a, b]$ system of continuous functions

$$\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots,$$

which is complete in the above defined sense. We know that if $f \in L_2'$ then for the numbers

$$c_k = (f, \varphi_k) \quad (k = 1, 2, \ldots) \tag{10}$$

Parseval's equality

$$\int_a^b f^2(x)\,dx = \sum_{k=1}^{\infty} c_k^2$$

is fulfilled. This is also true for the functions $f \in L_2$, only the integrals should be understood in the Lebesgue sense.

But the converse is also true: if the numbers c_k $(k = 1, 2, \ldots)$ are such that the series

$$\sum_{k=1}^{\infty} c_k^2 < \infty$$

converges, then in L_2 there exists a function $f(x)$ such that equalities (9) are fulfilled.

And in L_2' there may be no such function. This is how the imperfection of the L_2' space is manifested. There is an insufficient quantity of functions in this space for the converse to hold.

Sec. 4.10. Completeness of Trigonometric Functions

The tests for convergence of Fourier series were set forth in Sec. 4.4. We meant ordinary convergence there. Now we are going to formulate the test for convergence of a Fourier series in the mean square sense.

Let us denote by $L_{2*}' = L_{2*}'(-\pi, \pi)$ the class of all functions f of period 2π bounded on the interval $[-\pi, \pi]$ and continuous on it except possible for an infinite number

of points at which f has a discontinuity of the first kind.

If a funcion $f \in L'_{2*}$, *then its Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \tag{1}$$

$$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{\begin{matrix} \cos kt \\ \sin kt \end{matrix}\right\} dt \quad (k = 0,\ 1,\ 2,\ \ldots),$$

converges to it in the mean square, i.e.

$$\int_{-\pi}^{\pi} [f(x) - S_n(x)]^2\, dx \to 0 \quad (n \to \infty), \tag{2}$$

where

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

Formula (1) with an equality sign should be read in the present case as follows: the function $f(x)$ is the sum of its Fourier series convergent to it in the mean square on the interval $[-\pi, \pi]$.

Let us compute directly the integral in (2), taking into account the orthogonal properties of trigonometric functions:

$$\int_{-\pi}^{\pi} [f(x) - S_n(x)]^2\, dx$$

$$= \int_{-\pi}^{\pi} \left[f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right]^2 dx$$

$$= \int_{-\pi}^{\pi} f^2(x)\, dx - 2\frac{a_0}{2} \int_{-\pi}^{\pi} f(x)\, dx$$

$$- \sum_{k=1}^{n} \left(a_k \int_{-\pi}^{\pi} f(x) \cos kx\, dx + b_k \int_{-\pi}^{\pi} f(x) \sin kx\, dx \right) + \frac{a_0^2}{2}\pi$$

$$+ \pi \sum_{k=1}^{n} (a_k^2 + b_k^2) = \int_{-\pi}^{\pi} f^2(x)\, dx - \frac{\pi a_0^2}{2} - \pi \sum_{k=1}^{n} (a_k^2 + b_k^2).$$

For the function $f \in L_{2*}'$ this expression tends to zero as $n \to \infty$. But then there takes place the equality

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \qquad (f \in L_{2*}'), \qquad (3)$$

called Parseval's equality for trigonometric functions (or Liapunov's equality).

Remark 1. When comparing formula (3) with formula (7) from the preceding section, take into consideration that the latter was derived for an orthonormalized system, whereas formula (3) considered here is obtained for an orthogonal, but not normalized, system

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \ldots .$$

The functions

$$1, \quad \cos x, \quad \cos 2x, \quad \cos 3x, \quad \ldots \qquad (4)$$

form an orthogonal system on the interval $[0, \pi]$. The following theorem takes place.

Theorem 1. *Any function* $f \in L_2'(0, \pi)$, *i.e. piecewise continuous on* $[0, \pi]$ *can be expanded in a Fourier series of cosines*:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \qquad a_k = \frac{2}{\pi}\int_0^{\pi} f(t)\cos kt\,dt \qquad (5)$$

$$(k = 0, 1, 2, \ldots)$$

which is convergent to f *in the mean square on* $[0, \pi]$.

Indeed, this function can be extended to $[-\pi, \pi]$ in an even manner, and then periodically, with period 2π, to the entire real axis. As a result, a function $f \in L_{2*}'$ is obtained. Its Fourier serie sof functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$, has, by virtue of its evenness, exactly ferm (5), and, as we already know, this series converges to $f(x)$ in the mean square on $[-\pi, \pi]$, and the more so in the mean square on $[0, \pi]$.

The aforesaid can be expressed by the following words: the system of function (4) is orthogonal and is a complete system in $L_2'(0, n)$.

The following statement is also true: the system of functions

$$\sin x, \quad \sin 2x, \quad \sin 3x, \quad \ldots \tag{6}$$

is orthogonal and is a complete system in $L_2'(0, \pi)$, i.e. the following theorem takes place.

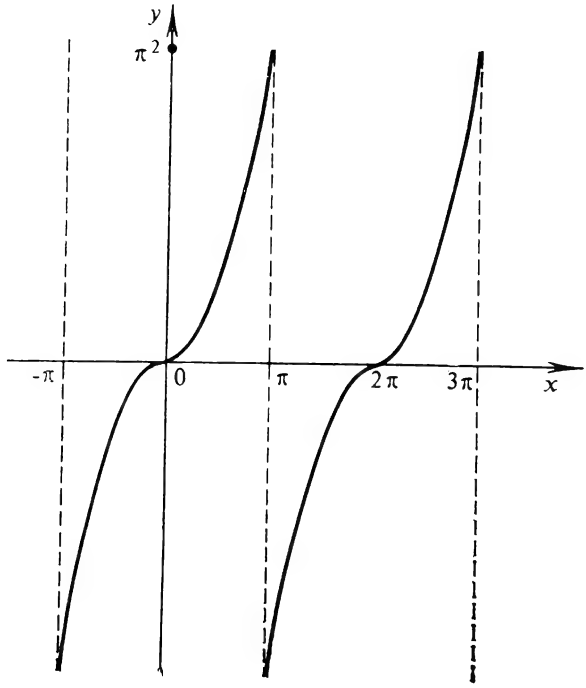

Fig. 118.

Theorem 2. *Any function* $f \in L_2'(0, \pi)$ *can be expanded in a Fourier series of sines*

$$f(x) = \sum_{k=1}^{\infty} b_k \sin kx, \qquad b_k = \frac{2}{\pi} \int_0^{\pi} f(t) \sin kt \, dt \tag{7}$$

$$(k = 1, 2, \ldots),$$

convergent to it in the mean square on $[0, \pi]$.

The orthogonality of system (6) is verified directly and, actually, follows from (2) of Sec. 4.5. As to the completeness, it follows from the following considerations.

Extend the function $f \in L_2'(0, \pi)$ to the interval $[-\pi, \pi]$ in an odd manner and then periodically with period 2π. Its Fourier series according to the system $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ converges in the mean square on $[-\pi, \pi]$, the more so on $[0, \pi]$. And this series has form (7).

Example 1. Expand the function $y = x^2$ $(0 \leqslant x \leqslant \pi)$ in a Fourier series of sines.

We extend this function in an odd manner to $[-\pi, 0]$ and then periodically with period 2π, to the entire real axis. Then the Fourier series of this function $\psi(x)$ will consist only of sines:

$$b_k = \frac{2}{\pi} \int_0^{\pi} x^2 \sin kx \, dx = \frac{2\pi (-1)^{k+1}}{k} + \frac{4}{\pi k^3} [(-1)^k - 1],$$

whence

$$b_{2k} = -\frac{\pi}{k}, \quad b_{2k-1} = \frac{2\pi}{2k-1} - \frac{8}{\pi (2k-1)^3}.$$

Thus,

$$\psi(x) = \sum_{k=1}^{\infty} b_k \sin kx.$$

The graph of the sum of this series is represented in Fig. 118.

Sec. 4.11. The Complex Form of Fourier Series

Let a_k and b_k be the Fourier coefficients of the function $f(x)$. On the basis of Euler's formulas

$$a_k \cos kx + b_k \sin kx = a_k \frac{e^{ikx} + e^{-ikx}}{2} + b_k \frac{e^{ikx} - e^{-ikx}}{2i} = c_k e^{ikx} + c_{-k} e^{-ikx},$$

where (letting $b_0 = 0$)

$$c_k = \frac{a_k - ib_k}{2}, \quad c_{-k} = \frac{a_k + ib_k}{2}. \tag{1}$$

Hence

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} (\cos kt - i \sin kt) f(t) \, dt = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} \, dt,$$

$$c_{-k} = \frac{1}{2\pi} \int_0^{2\pi} (\cos kt + i \sin kt) f(t) \, dt = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{ikt} \, dt.$$

These two equalities can be written in the form of a single formula

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-ikt}\, dt \quad (k = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

It is of importance to note that if $f(x)$ is a real function, then a_k and b_k are real, and the numbers c_k and c_{-k} being in general complex, are mutually conjugate:

$$c_{-k} = \bar{c}_k. \tag{3}$$

Obviously, the nth sum of Fourier's series of the function f can be written in the form

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) = \sum_{k=-n}^{n} c_k e^{ikx}, \tag{4}$$

and its Fourier series itself in the form of the series

$$f(x) \sim \frac{a_0}{2} + \sum_{1}^{\infty} (a_k \cos kx + b_k \sin kx) \sim \sum_{-\infty}^{\infty} c_k e^{ikx}. \tag{5}$$

We shall say that series (5) is convergent for a given value of x if there exists the limit

$$\lim_{n\to\infty} \sum_{-n}^{n} c_k e^{ikx} = \lim_{n\to\infty} S_n(x).$$

Thus, the above defined convergence is called a convergence in the principal-value sense.

We might regard the mentioned series as convergent if there exists the limit

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{-m}^{n} c_k e^{ikx},$$

when m and n increase without bound independently of each other.

The complex functions

$$\{e^{ikx}\} \quad (k = 0, \pm 1, \pm 2, \ldots) \tag{6}$$

form an orthogonal system on the interval $[0, 2\pi]$, since for $k \neq l$

$$(e^{ikx}, e^{ilx}) = \int_0^{2\pi} e^{ikx}\overline{e^{ilx}}\,dx = \int_0^{2\pi} e^{i(k-l)x}\,dx = \frac{e^{i(k-l)x}}{k-l}\Big|_0^{2\pi} = 0$$

(the first equality is written according to the definition of a scalar product for complex-valued functions, see remark 2 in Sec. 4.8). Further

$$(e^{ikx}, e^{ikx}) = \int_0^{2\pi} e^{ikx}e^{-ikx}\,dx = 2\pi.$$

Sec. 4.12. Notion of Fourier Integral. Iterated Fourier Integral

Let us first consider a piecewise smooth function $f(x)$ of period 2π satisfying the property

$$f(x) = \frac{f(x+0)+f(x-0)}{2}. \tag{1}$$

This means that f has period 2π, is continuous, and has a continuous derivative everywhere on the real axis except for a finite number of points on the period $[-\pi, \pi]$; and at these points there exist right-hand and left-hand limits of f and f'. Besides, we assume that equality (1) is fulfilled at any point. Of course, this condition is essential only for the points of discontinuity of f, since at points of continuity it is fulfilled automatically. The class of all indicated periodic functions f will be denoted by L'_*.

For each function $f \in L'_*$ there exists its Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{-\infty}^{+\infty} c_k e^{ikx}, \tag{2}$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt \qquad (k = 0, 1, 2, \ldots), \tag{3}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt \qquad (k = 1, 2, \ldots), \tag{4}$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt \qquad (k = 0, \pm 1, \pm 2, \ldots),$$
$$c_k = \frac{a_k - ib_k}{2}, \quad c_{-k} = \frac{a_k + ib_k}{2} \quad (k = 0, 1, 2, \ldots; b_0 = 0). \tag{5}$$

Let us also write the Nth sum of the Fourier series of the function f:

$$S_N^*(x) = \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx) = \sum_{k=-N}^{N} c_k e^{ikx}. \tag{6}$$

In the theory of Fourier series it is proved that for any function $f \in L_*'$ there takes place

$$\lim_{N\to\infty} S_N^*(x) = f(x), \tag{7}$$

i.e. the Fourier series of the function $f \in L_*'$ converges to it at any point x.

Fourier integrals can be introduced by analogy to Fourier series.

In this section we shall be concerned with nonperiodic functions f which are piecewise smooth and absolutely integrable over the real axis. Thus, for them the improper integral

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

is finite.

The term 'piecewise smooth function' is understood in the following sense. The function f is continuous and has a continuous derivative for all points x of the real axis with the exception of a finite number of points at which the function f or its derivative f' is discontinuous. But at the points of discontinuity there exist right-hand and left-hand limits both of f and f' and the following equality takes place:

$$f(x) = \frac{f(x+0) + f(x-0)}{2}.$$

The class of indicated nonperiodic functions will be denoted by $L' = L'(-\infty, \infty)$.

By analogy with the Fourier coefficients, for the functions $f \in L'$ the following functions are introduced:

$$a(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos ts \, dt \qquad (-\infty < s < \infty), \qquad (3')$$

$$b(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin st \, dt \qquad (-\infty < s < \infty), \qquad (4')$$

$$c(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ist} \, dt \qquad (-\infty < s < \infty). \qquad (5')$$

While the Fourier coefficients are defined for discrete values $k = 0, 1, 2, \ldots$, their analogues (3')-(5') are already functions of a continuous argument s.

By our assumption, the function f is piecewise continuous, nevertheless the functions $a(s)$, $b(s)$, and $c(s)$ are continuous.

For instance, let, for the sake of simplicity, the function f have only one point of discontinuity x_0. Then integral (3') can be split into two integrals

$$a(s) = \frac{1}{\pi} \int_{x_0}^{\infty} f(t) \cos ts \, dt + \frac{1}{\pi} \int_{-\infty}^{x_0} f(t) \cos ts \, dt. \qquad (8)$$

If we transform the function $f(t)$ at the point $t = x_0$ setting $f(x_0) = f(x_0 + 0)$, then a continuous function of s and t $(-\infty < s < \infty, x_0 < t < \infty)$ will be found under the first integral in (8). By Weierstrass' test (see Sec. 2.15, Theorem 3), the first integral is uniformly convergent, since

$$|f(t) \cos ts| \leqslant |f(t)|,$$

$$\int_{x_0}^{\infty} |f(t)| \, dt < \infty.$$

But then the first integral is a continuous function of s (see Sec. 2.15, Theorem 1). The continuity of the second integral in (8) with respect to s is proved in a similar way.

Let us also note that

$$\lim_{s\to\infty} a(s) = 0, \tag{9}$$

$$\lim_{s\to\infty} b(s) = 0, \tag{10}$$

$$\lim_{s\to\infty} c(s) = 0. \tag{11}$$

For instance, to prove property (9), let us introduce the substitution $t = u + \frac{\pi}{s}$ in integral (3′). Then

$$a(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(u+\frac{\pi}{s}\right)\cos(su+\pi)\,du$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} f\left(u+\frac{\pi}{s}\right)\cos us\,du$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} f\left(t+\frac{\pi}{s}\right)\cos ts\,dt.$$

This equality and (3′) yield:

$$|a(s)| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[f\left(t+\frac{\pi}{s}\right)-f(t)\right]\cos ts\,dt\right|$$

$$\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|f\left(t+\frac{\pi}{s}\right)-f(t)\right|dt \to 0 \qquad (s\to\infty).$$

The last relationship (tending to zero) should be proved, of course, but we are not going to do this, leaving the proof to the reader.

As an analogue of an individual term (harmonic) of a Fourier series it is natural to regard the function

$a(s)\cos xs + b(s)\sin xs$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left[\cos ts\cos xs + \sin st\sin xs\right]dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(t-x)\,s\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[e^{is(t-x)}+e^{-is(t-x)}\right]dt$$

$$= c(s)\,e^{isx} + c(-s)\,e^{-isx}, \tag{12}$$

or, to be more precise, the expression

$$(a(s)\cos xs + b(s)\sin xs)\,ds = (c(s)\,e^{ixs} + c(-s)\,e^{-ixs})\,ds. \tag{12'}$$

The expression

$$S_N(x) = \int_0^N [a(s)(\cos xs + b(s)\sin xs]\,ds$$

$$= \frac{1}{\pi}\int_0^N \int_{-\infty}^{\infty} f(t)\cos(t-x)\,s\,dt\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dt \int_0^N f(t)\cos(t-x)\,s\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_0^N \cos(t-x)\,s\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{\sin N(t-x)}{t-x}\,dt \tag{13}$$

is an analogue of the Nth Fourier sum (see (12)).

We have changed the order of integration. In the present case it is legitimate. On the basis of Fubini's* theorem, well known in mathematical analysis, we may interchange integrals in a multiple integral if this results in an absolutely convergent multiple integral. In the present case

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^N |f(t)\cos(t-x)\,s|\,dt\,ds$$

$$\leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^N |f(t)|\,ds\,dt = \frac{N}{\pi}\int_{-\infty}^{\infty} |f(t)|\,dt < \infty.$$

* Fubini, Guido (1879-1943). Italian mathematician.

By virtue of (12), the function $S_N(x)$ can also be written in a complex form

$$S_N(x) = \int_0^N [c(s)\, e^{isx} + c(-s)\, e^{-isx}]\, ds$$

$$= \int_{-N}^{N} c(s)\, e^{isx}\, ds = \int_{-N}^{N} e^{isx} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{isx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt\, ds. \qquad (13')$$

The function

$$S_N(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin N(t-x)}{t-x}\, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u+x) \frac{\sin Nu}{u}\, du \qquad (14)$$

is called a *simple Fourier integral.*

It is possible to prove that if $f \in L'(-\infty, \infty)$, then for any x

$$\lim_{N \to \infty} S_N(x) = f(x), \qquad (15)$$

i.e. there takes place the property analogous to property (7) for Fourier series.

Note that the Nth sum of a Fourier series of a periodic function can be written in the following way (see the explanations below):

$$S_N^*(x) = \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} f(t)\, dt$$

$$+ \sum_{k=1}^{N} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt\, dt \cos kx \right.$$

$$\left. + \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt\, dt \sin kx \right]$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^{N} (\cos kt \cos kx + \sin kt \sin kx) \right] dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^{N}{}' \cos k(t-x) \right] dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin (N+1/2)(t-x)}{2 \sin ((t-x)/2)} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) \frac{\sin (N+1/2) u}{2 \sin (u/2)} du. \tag{16}$$

In the last equality but one we have taken advantage of the formula (see our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 9.8, (15))

$$\frac{1}{2} + \sum_{k=1}^{N} \cos k\alpha = \frac{\sin (N+1/2)\alpha}{2 \sin (\alpha/2)}.$$

And in the last equality of (16) we have changed the variable: $t = u + x$. By virtue of this substitution the integral over the interval $[-\pi, \pi]$ with respect to t will turn into an integral over $[x - \pi, x + \pi]$ with respect to u, but the last interval may be again replaced by the interval $[-\pi, \pi]$ since the integrand function (of u) is of period 2π (see Sec. 4.3, (3)).

We see that the integral in the right-hand side of (16) is very much like integral (14). Therefore it is no wonder that both of them tend to the function $f(x)$ as $N \to \infty$.

From (13) and (15) it follows that

$$\lim_{N \to \infty} \frac{1}{\pi} \int_{0}^{N} ds \int_{-\infty}^{\infty} f(t) \cos (t-x) s \, dt = f(x). \tag{17}$$

Consequently, for any function $f \in L'(-\infty, \infty)$

$$\frac{1}{\pi} \int_{0}^{\infty} ds \int_{-\infty}^{\infty} f(t) \cos (t-x) s \, dt = f(x). \tag{18}$$

This is a very important equality which constitutes the base of the theory of Fourier series.

The double integral in (18) will be referred to as *Fourier's iterated integral* of the function f.

Equality (18) states that for the functions $f \in L'(-\infty, \infty)$ Fourier's iterated integral of f at a point x is equal to the value of the function f at the point x.

One cannot change the order of integration in (18). No good would come out of such a change. If we performed such a change, then we would have to integrate the function $\cos(t-x)s$ with respect to s (with t and x fixed) over an infinite interval $(-\infty, \infty)$, but such integral has no sense.

Hence, in iterated integral (18) we must first integrate the function $f(t)\cos(t-x)s$ with respect to t over $(-\infty, \infty)$, and then the same function with respect to s over $(0, \infty)$. Both integrals will be improper. The integral with respect to t is, obviously, absolutely convergent; as to the integral with respect to s, this is, generally speaking, not so.

Of course, we may replace the letters s and t in integral (18) by any other letters s', t' which will not change the value of this integral.

From (13) and (15) we obtained formula (18). On the other hand, from (13′) and (15) we shall get another important formula which is true for the functions $f \in L'$ $(-\infty, \infty)$:

$$\lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{isx}\,ds \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(t)\,e^{-ist}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\,ds \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\,e^{-ist}\,dt = f(x). \quad (19)$$

Let us introduce the following notation:

$$\tilde{f}(x) = \lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(t)\,e^{-ixt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\,e^{-ixt}\,dt = \sqrt{2\pi}\,c(x),$$

$$\hat{f}(x) = \lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(t)\,e^{ixt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\,e^{ixt}\,dt = \sqrt{2\pi}\,c(-x).$$

Here $\tilde{f}(x)$ is called Fourier's transformation or Fourier's direct transformation of the function f, and $\hat{f}(x)$ is Fourier's inverse transformation of the function f. The two operations symbolized by the signs "$\sim$" and "$\frown$" are mutually inverse. If the operation "$\sim$" is performed on f and operation "$\frown$" is then performed on the obtained function $\tilde{f}$, then, as is seen from (19), the original function f is obtained once again:

$$\hat{\tilde{f}} = f.$$

Problem 1. Prove the following formulas for the functions $f \in L'(-\infty, \infty)$:

(1) $\hat{f}(-t) = \tilde{f}(t)$;

(2) $\tilde{f}(-x) = \hat{f}(x)$;

(3) $\widetilde{f(at)} = \frac{1}{|a|}\tilde{f}(x/a)$;

(4) $\widehat{f(at)} \frac{1}{|a|}\hat{f}\left(\frac{x}{a}\right)(a \neq 0)$;

(5) $\widehat{e^{i\mu t}\tilde{f}} = \widetilde{e^{-i\mu t}\hat{f}} = f(x+\mu)$ (μ real);

(6) $\tilde{\hat{f}} = f$.

For instance,

$$\widehat{e^{i\mu t}\tilde{f}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\mu t}e^{ixt}\,dt\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-iut}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(\mu+x)t}\,dt\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-iut}\,du$$

$$= \hat{\tilde{f}}(\mu+x) = f(\mu+x).$$

Sec. 4.13. Fourier Cosine and Sine Transforms

By virtue of (18) from the preceding section there takes place for $f \in L'(-\infty, \infty)$ the equality

$$f(x)=\frac{1}{\pi}\int_0^\infty ds \int_{-\infty}^\infty f(t)\cos(t-x)\,s\,dt$$

$$=\frac{1}{\pi}\int_0^\infty \cos xs\,ds \int_{-\infty}^\infty f(t)\cos ts\,dt$$

$$+\frac{1}{\pi}\int_0^\infty \sin xs\,ds \int_{-\infty}^\infty f(t)\sin ts\,dt. \qquad (1)$$

If the function $f(t)$ is even, then the second integral in the right-hand side of (1) is zero and in the first one the integration with respect to t over $(-\infty, \infty)$ is reduced to integration over $(0, \infty)$ and we shall obtain the formula

$$\frac{2}{\pi}\int_0^\infty \cos xs\,ds \int_0^\infty f(t)\cos ts\,dt = f(x). \qquad (2)$$

And for an odd function $f(t)$ the first integral in the right-hand side of (1) is equal to zero, and the function $f(t)\sin ts$ is even. Therefore

$$\frac{2}{\pi}\int_0^\infty \sin xs\,ds \int_0^\infty f(t)\sin ts\,dt = f(x). \qquad (3)$$

In formulas (2) and (3) we may assume that $x \geqslant 0$ and $f(t)$ is an arbitrary piecewise smooth function belonging to $L'(0, \infty)$. In fact, we use only the values of f on the semi-axis $[0, \infty)$ in these formulas. Let us explain this note in more detail.

Let there be given a piecewise smooth function $f \in L'(0, \infty)$ such that $f(0) = f(0+0)$. Extending it to the entire real axis in an even manner, we obtain an even piecewise smooth function $f \in L'(-\infty, \infty)$ for which formula (2) holds true; in particular, it is valid for $x \geqslant 0$.

Let us now assume that for our piecewise smooth function $f \in L'(0, \infty)$ the equality $f(0) = 0$ is fulfilled (in general,

$f(0+0) \neq f(0)$). Extending f in an odd manner to $(-\infty, \infty)$, we obtain an odd piecewise smooth function $f \in L'(-\infty, \infty)$, for which formula (3) holds true; in particular, it is valid for $x \geqslant 0$. Note that in formula (3) $f(0) = 0$, whereas in formula (2) the value $f(0) = f(0+0)$ may be arbitrary.

The integrals

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos ts \, dt, \quad \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin ts \, dt$$

are called, respectively, *Fourier's cosine transform* and *Fourier's sine transform* of the function f. It follows immediately from formulas (2) and (3) that if Fourier's cosine or sine transformation (i.e. the operation of forming the first or the second integral written above) is performed twice in a consecutive order on a given piecewise smooth function $f \in L'(0, \infty)$, the result is again the original function f. In this sense Fourier's sine and cosine transformations are inverse to themselves.

Sec. 4.14. Examples

The following equalities hold true (the explanations are given below):

(1) $$f(x) = \begin{cases} 1, & 0 \leqslant |x| < a \\ 0, & |x| > a \end{cases} = \frac{2}{\pi} \int_0^\infty \cos xs \frac{\sin as}{s} \, ds.$$

(2) $$f(x) = \begin{cases} \operatorname{sgn} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases} = \frac{2}{\pi} \int_0^\infty \sin xs \frac{1 - \cos s}{s} \, ds.$$

(3) $$f(x) = \begin{cases} 1, & a < x < b \\ 0, & x < a, \ b < x \end{cases} = \frac{1}{\pi} \int_0^\infty \frac{\sin s(x-a) - \sin s(x-b)}{s} \, ds.$$

(4) $$\int_0^\infty e^{-as} \cos sx \, ds = \frac{a}{a^2 + x^2} \quad (a > 0).$$

(5) $$\int_0^\infty e^{-as} \sin sx \, ds = \frac{x}{a^2 + x^2} \quad (a > 0).$$

$$(6)\quad e^{-as}=\frac{2a}{\pi}\int_0^\infty \frac{\cos xs}{a^2+x^2}\,dx \quad (a>0,\ 0\leqslant s<\infty).$$

$$(7)\quad e^{-as}=\frac{2}{\pi}\int_0^\infty \frac{x}{a^2+x^2}\sin xs\,dx \quad (a>0,\ 0<s<\infty).$$

$$(8)\quad f(x)=\begin{cases}\sin x, & |x|\leqslant\pi\\ 0 & |x|>\pi\end{cases}=\frac{2}{\pi}\int_0^\infty \frac{\sin s\pi}{1-s^2}\sin sx\,ds.$$

$$(9)\quad f(x)=\begin{cases}\cos x, & |x|\leqslant\frac{\pi}{2}\\ 0, & |x|>\frac{\pi}{2}\end{cases}=\frac{2}{\pi}\int_0^\infty \frac{\cos\frac{s\pi}{2}}{1-s^2}\cos sx\,ds.$$

$$(10)\quad f(x)=e^{-\alpha|x|}\cos\beta x$$
$$=\frac{\alpha}{\pi}\int_0^\infty \cos sx\left[\frac{1}{(s-\beta)^2+\alpha^2}+\frac{1}{(s+\beta)^2+\alpha^2}\right]ds\ (\alpha>0).$$

$$(11)\quad f(x)=e^{-\alpha|x|}\sin\beta x$$
$$=\frac{4\alpha\beta}{\pi}\int_0^\infty \frac{s\sin sx\,ds}{[(s-\beta)^2+\alpha^2]\,[(s+\beta)^2+\alpha^2]}\ (\alpha>0).$$

Using the ordinary methods of the theory of indefinite integrals it is not clear how to compute the integrals standing in the right-hand sides of equalities (1)-(3). On the other hand, the functions in (1)-(3) are piecewise smooth and belong to $L'(-\infty, \infty)$ ($f\in L'(-\infty, \infty)$). Therefore, formula (1) from the preceding section is applicable to them. This formula is simplified and has form (2) (Sec. 4.13) if f is an even function, and if f is odd, then the formula has form (3) (Sec. 4.13). For instance, function (1) is even and therefore

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos xs\,ds\int_0^a \cos ts\,dt=\frac{2}{\pi}\int_0^\infty \cos xs\,\frac{\sin sa}{s}\,ds,$$

where it should be assumed that at the points of discontinuity of f the following equality is fulfilled:

$$f(x)=\frac{1}{2}[f(x+0)+f(x-0)].$$

Integrals (4) and (5) are evaluated by integrating by parts. Using equality (4), we have

$$\frac{2a}{\pi}\int_0^\infty \frac{\cos xs}{a^2+x^2}\,dx$$

$$=\frac{2}{\pi}\int_0^\infty \cos xs\,ds\int_0^\infty e^{-a\lambda}\cos\lambda x\,d\lambda = e^{-a|s|},$$

where the last equality takes place by virtue of formula (2) of Sec. 4.13 which is applicable since $e^{-a\lambda}\in L'(0,\infty)$ is a smooth function. Thus, equality (6) has been proved.

Reasoning in a similar way, we obtain formula (7) from (5) with the aid of formula (3) (Sec. 4.13).

Function (8) is odd piecewise smooth. To obtain the required integral, we represent this function by formula (3) (Sec. 4.13), where the inner integral is equal to

$$\int_0^\infty \sin st\,f(t)\,dt = \int_0^\infty \sin st\sin t\,dt$$

$$=\frac{1}{2}\int_0^\pi [\cos t(s-1)-\cos t(s+1)]\,dt$$

$$=\frac{1}{2}\left[\frac{\sin\pi(s-1)}{s-1}-\frac{\sin\pi(s+1)}{s+1}\right]=\frac{\sin\pi s}{1-s^2}.$$

The representation of function (9) is obtained in a similar way with the aid of formula (2) from Sec. 4.13.

Function (10) is even. To obtain the desired integral, we represent it by formula (2) from Sec. 4.13, where the inner integral is equal to

$$\int_0^\infty e^{-\alpha t}\cos\beta t\cos st\,dt$$

$$=\frac{1}{2}\int_0^\infty e^{-\alpha t}\cos(\beta+s)t\,dt+\frac{1}{2}\int_0^\infty e^{-\alpha t}\cos(\beta-s)t\,dt$$

$$=\frac{\alpha}{2}\left[\frac{1}{(\beta+s)^2+\alpha^2}+\frac{1}{(\beta-s)^2+\alpha^2}\right].$$

The last equality is written by virtue of (4).

Analogous reasoning holds for function (11) when using formula (3) from the preceding section.

Finally, let us give one more example which is computed by using quite different techniques.

(12) Find Fourier's cosine transform of the function $\exp(-t^2)$.

Let

$$\int_0^\infty \exp(-\lambda^2)\cos\lambda s\, d\lambda = I(s).$$

It is easily seen that (see Sec. 2.13, Example 3)

$$I(0) = \int_0^\infty \exp(-\lambda^2)\, d\lambda = \frac{1}{2}\sqrt{\pi}.$$

Differentiating the function $I(s)$, we obtain

$$I'(s) = -\int_0^\infty \lambda \exp(-\lambda^2)\sin\lambda s\, d\lambda$$

(differentiation is legitimate here since the last integral is uniformly convergent). In integrating by parts, the derivative $I'(s)$ can be represented in the form ($\lambda \exp(-\lambda^2)\, d\lambda = dv$, $\sin\lambda s = u$)

$$I'(s) = \frac{\sin\lambda s}{2}\exp(-\lambda^2)\Big|_{\lambda=0}^{\infty} - \frac{s}{2}\int_0^\infty \exp(-\lambda^2)\cos\lambda s\, d\lambda$$

$$= -\frac{s}{2} I(s).$$

Solving the last first-order differential equation, we have

$$\frac{dI}{I} = -\frac{s}{2}ds, \quad \ln\left|\frac{I}{C}\right| = -\frac{s^2}{4}, \quad I(s) = C\exp\left(-\frac{s^2}{4}\right).$$

From the condition $I(0) = \sqrt{\pi}/2$ we find that $C = \sqrt{\pi}/2$. Thus,

$$\int_0^\infty \exp(-\lambda^2)\cos\lambda s\, d\lambda = \frac{\sqrt{\pi}}{2}\exp\left(-\frac{s^2}{4}\right).$$

Sec. 4.15. Approximation of Fourier Integral

Let us explain the physical aspect of the notion of Fourier's integral by considering a nonperiodic motion in which the ordinate y of some point is a function $y = f(x)$ of time x.

The function $f(x)$ can be written in the following way:

$$f(x) = \int_0^\infty [a(s)\cos xs + b(s)\sin xs]\,ds$$

$$= \int_0^\infty [c(s)\,e^{-isx} + c(-s)\,e^{isx}]\,ds.$$

For sufficiently large N and then for sufficiently small Δs ($\Delta s = \Delta s_j$) we obtain approximately

$$f(x) \sim \int_0^N [a(s)\cos xs + b(s)\sin xs]\,ds$$

$$= \int_0^N [c(s)\,e^{-isx} + c(-s)\,e^{isx}]\,ds$$

$$\sim \sum_j [a(s_j)\cos xs_j + b(s_j)\sin xs_j]\,\Delta s$$

$$= \sum_j [c(s_j)\,e^{-ixs_j} + c(-s_j)\,e^{ixs_j}]\,\Delta s. \qquad (1)$$

The first approximation can be realized with any accuracy in any case if the integrals of $a(s)$ and $b(s)$ (and, consequently, of $c(s)$) are absolutely convergent on $(0, \infty)$, in particular, if the functions $a(s)$, $b(s)$ (and, consequently, $c(s)$) are equal to zero for $s > s_0$, where s_0 is some number. The second approximation can be obtained in any case for the values of x belonging to an arbitrarily given interval $[x_1, x_2]$. And (for a given interval $[x_1, x_2]$) it is possible to choose the necessary numbers s_j so that they turn out to be rational. But then the motion $y = f(x)$ will be approximated on the time interval $[x_1, x_2]$ by the sum of harmonic oscillations, even with a common period.

The *spectrum of a periodic function* $f(x)$ is defined as the set of all its Fourier coefficients. In particular, it is seen

from the spectrum what nontrivial (not equal identically to zero) harmonics the periodic motion $y = f(x)$ consists of.

The *spectrum of a nonperiodic function* $f(x)$ is known as the functions $a(s)$ and $b(s)$ or the function $c(s)$ generated by this function.

If $a(s)$ and $b(s)$ are equal to zero outside the interval (p, q), then the sum approximating $f(x)$ by formula (1) consists of harmonic oscillations with frequencies $s_j \in (p, q)$.

The function $\tilde{f}(s) = \sqrt{2\pi} c(s)$ is also called the spectrum of f.

CHAPTER 5

EQUATIONS OF MATHEMATICAL PHYSICS

Sec. 5.1. Temperature of a Body

Let us consider a physical body Ω. Its temperature at the point (x, y, z) at an instant of time t will be denoted by

$$u = u(x, y, z, t).$$

Let us show that the function u satisfies the differential equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad ((x, y, z) \in \Omega) \tag{1}$$

or, using the notation

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

the equation

$$\frac{\partial u}{\partial t} = a^2 \Delta u, \tag{1'}$$

which is called the *heat conduction equation.* This is an example of a second-order linear partial differential equation.

Consider an elementary cube σ in the body Ω (Fig. 119). The quantity of heat passing through the left-hand face of σ in the direction from right to left during the interval of time $(t, t + \Delta t)$ is equal, with the accuracy to infinitesimals of a higher order, to

$$\alpha \frac{\partial u}{\partial x}(x, y, z, t)\, \Delta y\, \Delta z\, \Delta t.$$

Here α is thermal conductivity of the body which is assumed to be constant at any of its points. The thing is that the indicated quantity of heat is, obviously, proportional to the number α, area $\Delta y\, \Delta z$ of the face under consideration, time increment Δt, and to the rate of temperature change

in the direction of the x-axis which is equal to the partial derivative $\frac{\partial u}{\partial x}$. The latter varies within the limits of the face, but, neglecting the infinitesimals of a higher order, we may assume that it is equal to $\frac{\partial u}{\partial x}$ at every point (x, y, z, t) belonging to this face.

The quantity of heat passing through the right-hand face of σ in the direction from right to left is, obviously, equal to

$$\alpha \frac{\partial u}{\partial x}(x+\Delta x,\ y,\ z,\ t)\,\Delta y\,\Delta z\,\Delta t.$$

And the quantity of heat which entered the cube σ through its left-hand and right-hand faces during the indicated interval of time is equal to

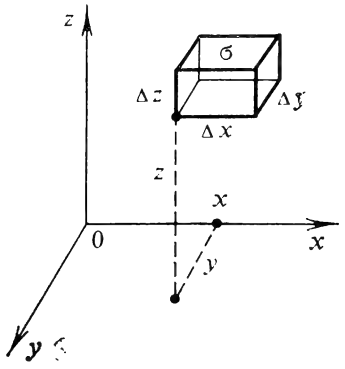

Fig. 119.

$$\alpha \frac{\partial u}{\partial x}(x+\Delta x,\ y,\ z,\ t)\,\Delta y\,\Delta z\,\Delta t$$

$$-\alpha \frac{\partial u}{\partial x}(x,\ y,\ z,\ t)\,\Delta y\,\Delta z\,\Delta t$$

$$\sim \alpha \frac{\partial^2 u}{\partial x^2}(x,\ y,\ z,\ t)\,\Delta x\,\Delta y\,\Delta z\,\Delta t.$$

The total quantity of heat which entered σ during the time $(t,\ t+\Delta t)$ is, obviously, equal to the sum of quantities of heat which entered σ through all of its faces during this time:

$$\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\Delta x\,\Delta y\,\Delta z\,\Delta t. \qquad (2)$$

But this number (quantity of heat) is also equal to

$$\beta \frac{\partial u}{\partial t}\Delta x\,\Delta y\,\Delta z\,\Delta t, \qquad (3)$$

where β is heat capacity of the body which is regarded as constant at all of its points.

Equating (2) to (3), after reductions we obtain differential equation (1), where

$$a^2=\frac{\alpha}{\beta}.$$

Thus, we have shown that temperature of the body Ω is the function $u = u(x, y, z, t)$ satisfying equation (1), where a^2 is a positive constant whose physical meaning was explained above. However, we have confined ourselves to the case when the body has a constant heat capacity and a constant thermal conductivity.

Differential equation (1) has an infinite number of solutions. To find among them a definite solution, we have to impose some additional conditions on the function u. These are usually initial and boundary conditions.

In the sections below we shall consider several mathematical problems connected with this question.

Sec. 5.2. Dirichlet Problem

Heat distribution in a body is called stationary if its temperature depends on the position of the point (x, y, z), but is independent of the time t, i.e.

$$u = u(x, y, z).$$

In this case

$$\frac{\partial u}{\partial t} = 0$$

and the function u satisfies the equation

$$\Delta u = 0.$$

Definition. A function $u(x, y, z)$ is called *harmonic* in a domain Ω if it possesses continuous partial derivatives of the second order in Ω and satisfies the equation

$$\Delta u = 0. \tag{1}$$

Equation (1) is called Laplace's* equation. The following theorem holds true.

Theorem 1. *Let a bounded domain Ω in space have a piecewise smooth boundary (of a surface) Γ in which a continuous function $f(x, y, z)$ is specified. Then there exists in the closure $\overline{\Omega}$ a unique continuous function $u(x, y, z)$, harmonic in Ω, such that*

$$u|_\Gamma = f(x, y, z).$$

* Laplace, Pierre Simon (1749-1827). Outstanding French mathematician, astronomer, and physicist.

Theorem 1 has an obvious physical interpretation. If on the boundary Γ of the body Ω all the time a constant temperature u is maintained which is equal to $u|_\Gamma = f(x, y, z)$, where $f(x, y, z)$ is a given continuous (on Γ) function, then inside the body quite definite (unique) temperature $u(x, y, z)$ will set in. This assertion should be regarded as obvious from the physical point of view. But it can be proved mathematically as well. This problem, called the *Dirichlet problem*, is investigated very well and various approximate methods of its solution are given.

The Dirichlet problem has also a great practical application in the two-dimensional case. In the latter case it is formulated as follows.

A continuous function $f(x, y)$ is defined on a piecewise smooth boundary Γ of a plane domain Ω. It is required to find the function $u(x, y)$, continuous on $\overline{\Omega} = \Omega + \Gamma$ and harmonic in Ω, i.e. having second continuous partial derivatives and satisfying Laplace's equation in Ω:

$$\Delta u = 0 \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

This problem is solved in a positive way: in $\overline{\Omega}$ there exists a unique function $u(x, y)$ satisfying the requirements of this problem.

Of particular importance are the cases when the Dirichlet problem is solved effectively.

Given below is an effective solution of the Dirichlet problem for a circle.

Sec. 5.3. Dirichlet Problem for a Circle

Theorem 1. *Let σ be an open unit circle with centre at the origin of the rectangular coordinate system (x, y) and on its boundary we are given a continuous periodic (of period 2π) function $f(\theta)$, where θ is a polar angle of a point on Γ. Then in the closure $\overline{\sigma} = \sigma + \Gamma$ of a circle σ there exists a unique function $u(x, y)$, continuous in $\overline{\sigma}$, harmonic in σ and equal to $f(\theta)$ on Γ:*

$$u|_\Gamma = f(\theta). \tag{1}$$

In polar coordinates (ρ, θ) *the function* $u = u(\rho, \theta)$ *is written in the form of the series*

$$u(\rho, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \rho^k (a_k \cos k\theta + b_k \sin k\theta), \qquad (2)$$

where

$$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \left\{\begin{matrix} \cos k\theta \\ \sin k\theta \end{matrix}\right\} d\theta \quad (k = 0, 1, 2, \ldots)$$

are the Fourier coefficients of the function $f(\theta)$.

We shall prove Theorem 1 under the supposition that the function $f(\theta)$ has a second continuous derivative, although the theorem holds if $f(\theta)$ is simply continuous.

Let us expand the function $f(\theta)$ into a Fourier series

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta).$$

Since $f(\theta)$ has a second continuous derivative, then

$$|a_k| \leqslant \frac{2M}{k^2}, \quad |b_k| \leqslant \frac{2M}{k^2}, \quad k \neq 0, \qquad (3)$$

where the constant $M = \max\limits_{-\pi \leqslant t \leqslant \pi} |f''(t)|$ (see Sec. 4.7). We have

$$|\rho^k (a_k \cos k\theta + b_k \sin k\theta)| \leqslant |a_k| + |b_k| \quad (0 \leqslant \rho \leqslant 1),$$

and since the series

$$\left|\frac{a_0}{2}\right| + \sum_{k=1}^{\infty} (|a_k| + |b_k|) \leqslant \frac{|a_0|}{2} + 4M \sum_{k=1}^{\infty} \frac{1}{k^2}$$

is convergent, then, by Weierstrass' theorem, series (2) is uniformly convergent on $\bar{\sigma}$. But then $u(\rho, \theta)$ is a continuous on $\bar{\sigma}$ function as a sum of a uniformly convergent series of continuous functions. In addition

$$u(1, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta) = f(\theta),$$

i.e. property (1) is fulfilled.

Each term of series (2) satisfies Laplace's equation in polar coordinates

$$\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} = 0$$

(see our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 9.9, Example 3). Besides, the following equalities take place $(0 \leqslant \rho < 1)$:

$$\left.\begin{aligned} \frac{\partial u}{\partial \rho} &= \sum_{k=1}^{\infty} k\rho^{k-1}(a_k \cos k\theta + b_k \sin k\theta), \\ \frac{\partial^2 u}{\partial \rho^2} &= \sum_{k=2}^{\infty} k(k-1)\rho^{k-2}(a_k \cos k\theta + b_k \sin k\theta), \\ \frac{\partial^2 u}{\partial \theta^2} &= \sum_{k=1}^{\infty} k^2\rho^k(-a_k \cos k\theta - b_k \sin k\theta). \end{aligned}\right\} \qquad (4)$$

In this case termwise differentiation of series (2) is legitimate, since for any positive $\rho_0 < 1$ the terms of series (4) do not exceed the number

$$\rho_0^k k^2 \frac{4M}{k^2} = 4M\rho_0^k \quad (0 \leqslant \rho \leqslant \rho_0),$$

and the series

$$4M \sum_{1}^{\infty} \rho_0^k < \infty \quad (0 < \rho_0 < 1),$$

is convergent.

Therefore the sum of series (2) $u(\rho, \theta)$ is the solution of the problem in question (is a harmonic function).

We are not going to prove that the solution of the Dirichlet problem is unique.

Sec. 5.4. Dirichlet Problem for a Half-plane

Let in the half-plane $R_2^+ = \{-\infty < x < \infty,\ y > 0\}$ it be required to find the bounded solution of Laplace's equation

$$\Delta u = 0, \qquad (1)$$

satisfying the boundary condition

$$u(x, 0) = \varphi(x) \quad (-\infty < x < \infty). \tag{2}$$

It is easy to check that the functions

$$u_\lambda(x, y) = [\alpha(\lambda) \cos \lambda x + \beta(\lambda) \sin \lambda x] \exp(-\lambda y)$$

are bounded and harmonic on R_2^+ for any fixed $\lambda > 0$, i.e. satisfy equation (1) on R_2^+. But then the sum of such functions and even the integral with respect to the parameter will also be a solution of equation (1):

$$u(x, y) = \int_0^\infty [\alpha(\lambda) \cos \lambda x + \beta(\lambda) \sin \lambda x] \exp(-\lambda y)\, d\lambda, \tag{3}$$

if only the differentiation under the integral sign with respect to the parameters x and y is legitimate. The constants $\alpha(\lambda)$ and $\beta(\lambda)$ are found from condition (2)

$$u(x, 0) = \int_0^\infty [\alpha(\lambda) \cos \lambda x + \beta(\lambda) \sin \lambda x]\, d\lambda = \varphi(x). \tag{4}$$

Let us write the expansion of the function $\varphi(x)$ in the Fourier integral

$$\varphi(x) = \int_0^\infty \left\{ \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\xi) \cos \xi\lambda\, d\xi \right) \cos \lambda x + \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\xi) \sin \xi\lambda\, d\xi \right) \sin \lambda x \right\} d\lambda. \tag{5}$$

Comparing formulas (4) and (5), we see that

$$\alpha(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\xi) \cos \xi\lambda\, d\xi, \quad \beta(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\xi) \sin \xi\lambda\, d\xi. \tag{6}$$

Substituting these functions into (3), we obtain

$$u(x,\ y)=\frac{1}{\pi}\int_0^{\infty}\exp(-\lambda y)\left[\int_{-\infty}^{\infty}\varphi(t)\cos\lambda(t-x)\,dt\right]d\lambda$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)\left[\int_0^{\infty}\exp(-\lambda y)\cos\lambda(t-x)\,d\lambda\right]dt.$$

According to Sec. 4.14, Example 4, we have

$$u(x,\ y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(t)\,\frac{y\,dt}{y^2+(t-x)^2}. \tag{3'}$$

Remark 1. Let the function $\varphi(x)$ have continuous derivatives up to the fourth order inclusive and satisfy the conditions

$$\varphi^{(k)}(x)\to 0,\quad x\to\infty\quad (k=0,\ 1,\ 2,\ 3,\ 4);$$

$$M_k=\frac{1}{\pi}\int_{-\infty}^{\infty}|\varphi^{(k)}(x)|\,dx<\infty\quad (k=0,\ 1,\ 2,\ 3,\ 4).$$

Then from (6) there follow the estimates

$$|\alpha(\lambda)|\leqslant\frac{c}{1+\lambda^4},\quad |\beta(\lambda)|\leqslant\frac{c}{1+\lambda^4}, \tag{7}$$

where c is some constant. Indeed, if $|\lambda|<1$, then

$$|\alpha(\lambda)|\leqslant\frac{1}{\pi}\int_{-\infty}^{\infty}|\varphi(\xi)|\,d\xi=M_0\leqslant\frac{2M_0}{1+\lambda^4}. \tag{8}$$

But if $|\lambda|\geqslant 1$, then, integrating by parts four times, we obtain

$$\alpha(\lambda)=\frac{1}{\pi}\varphi(\xi)\frac{\sin\lambda\xi}{\lambda}\Big|_{-\infty}^{\infty}-\frac{1}{\pi\lambda}\int_{-\infty}^{\infty}\varphi'(\xi)\sin\xi\lambda\,d\xi$$

$$=-\frac{1}{\pi\lambda}\int_{-\infty}^{\infty}\varphi'(\xi)\sin\lambda\xi\,d\xi=\ldots=\frac{1}{\pi\lambda^4}\int_{-\infty}^{\infty}\varphi^{(4)}(\xi)\cos\xi\lambda\,d\xi,$$

whence

$$|\alpha(\lambda)| \leqslant \frac{M_4}{\lambda^4} \leqslant \frac{2M_4}{1+\lambda^4}. \tag{9}$$

The first inequality of (7) follows from (8) and (9), the second is proved in a similar way.

Estimates (7) ensure existence, continuity, and boundedness of the functions u, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ in the upper half-plane R_2^+.

Remark 2. It is possible to prove that if the function $\varphi(x)$ is continuous and bounded on $(-\infty, \infty)$, then the solution of the Dirichlet problem obtained with the aid of formula (3′) for the upper half-plane is unique in the class of bounded functions.

Sec. 5.5. Equation of Heat Conduction in a Rod

Consider a thin insulated (coated with thermal insulation) rod lying on the interval $[0, \pi]$ of the x-axis (Fig. 120). It is supposed that its physical properties are the same at points belonging to any of its cross sections. Therefore the temperature of the rod is the function

$$u = u(x, t)$$

of the abscissa x of a cross section and time t.

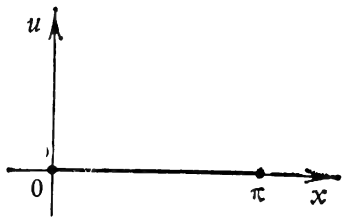

Fig. 120.

On the basis of what was said in Sec. 5.1 the function u satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $a^2 > 0$ is a constant if we assume that heat capacity and heat conduction of the rod are independent of x.

Let us formulate the following problem: find the function $u(x, t)$ continuous for $t \geqslant 0$, $0 \leqslant x \leqslant \pi$, having the continuous partial derivatives $\frac{\partial^2 u}{dx^2}$ and $\frac{\partial u}{\partial t}$ for $t > 0$, $0 \leqslant x \leqslant \pi$, satisfying differential equation (1) for $t > 0$, $0 \leqslant x \leqslant \pi$, and the following conditions:

(1) the initial condition

$$u(x, 0) = f(x) \quad (0 \leqslant x \leqslant \pi), \tag{2}$$

where $f(x)$ is a continuous function defined on the interval $[0, \pi]$;

(2) the boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad \forall t \geqslant 0. \tag{3}$$

Thus, it is supposed that at the zero time $t = 0$ the temperature in the rod is expressed by the function $f(x)$ (see (2)) and during the whole time of the experiment a zero temperature is artificially maintained at the ends of the rod (see (3)).

We shall solve equation (1) using *Fourier's method of separating the variables.* Its essence consists in that we find out the particular solutions of equation (1) satisfying for the time being only boundary conditions (3), in the form of a product of two functions each of which depends only on one variable:

$$u(x, t) = X(x)\, T(t). \tag{4}$$

Here we seek only for nontrivial solutions, i.e. for ones not equal identically to zero. From (4) we have

$$\frac{\partial^2 u}{\partial x^2} = T(t)\frac{d^2X}{dx^2} = TX'', \qquad \frac{\partial u}{\partial t} = XT'.$$

Substituting these expressions into (1), we obtain

$$\frac{1}{a^2}\frac{T'}{T} = \frac{X''}{X}. \tag{5}$$

The left-hand member of (5) is independent of x, its right-hand member being independent of t, therefore

$$\frac{1}{a^2}\frac{T'}{T} = \frac{X''}{X} = -\mu, \tag{6}$$

where μ is a constant.

Thus, the functions $X(x)$ and $T(t)$ satisfy the ordinary differential equations

$$X'' + \mu X = 0, \tag{7}$$

$$T' + a^2\mu T = 0, \tag{8}$$

where μ is some constant.

Since we look for the solutions satisfying (3), then for all t's there must be fulfilled the equality

$$u(0, t) = X(0) T(t) = 0, \quad u(\pi, t) = X(\pi) T(t) = 0.$$

If we assume that $T(t) = 0$, $\forall t$, then $u(x, t) \equiv 0$ for all x's and t's. Therefore there is at least one t for which $T(t) \neq 0$. But then

$$X(0) = X(\pi) = 0. \tag{9}$$

We have come to the following problem. It is required to find such numbers μ for which differential equation (7) has a nontrivial (not equal identically to zero) solution on the interval $[0, \pi]$ satisfying boundary conditions (9).

This problem is known as the *Sturm-Liouville* problem* for equation (7) on the interval $[0, \pi]$ under boundary conditions (9). The desired numbers μ are called the *eigenvalues* of the Sturm-Liouville problem, and the corresponding nontrivial functions satisfying boundary conditions (9) are termed the *eigenfunctions* corresponding to these eigenvalues.

We shall seek the solution of the above stated problem among the positive numbers $\mu = \lambda^2 > 0$ ($\lambda > 0$). In this case the numbers $\pm i\lambda$ are the roots of the characteristic equation, therefore the general solution of equation (7) will be written as follows:

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x.$$

From (9) we find

$$\left.\begin{aligned} 0 &= C_1, \\ 0 &= C_1 \cos \lambda\pi + C_2 \sin \lambda\pi, \end{aligned}\right\} \quad \text{or} \quad \begin{aligned} C_1 &= 0 \\ C_2 \sin \lambda\pi &= 0. \end{aligned}$$

To obtain a solution not equal identically to zero, we must assume that $C_2 \neq 0$ and $\sin \lambda\pi = 0$. The latter is possible only for natural $\lambda = k = 1, 2, 3, \ldots$. To every k there corresponds a solution

$$X_k(x) = d_k \sin kx \quad (k = 1, 2, \ldots),$$

---

* Sturm, Jacques Charles Francois (1803-1855). Swiss-French analyst and mathematical physicist.

Liouville, Joseph (1809-1882). French mathematician.

satisfying, obviously, boundary conditions (9). This is a nontrivial (not equal identically to zero) solution of equation (7). Thus, the numbers

$$\mu_k = k^2 \quad (k = 1, 2, 3, \ldots)$$

are the eigenvalues of the above formulated boundary problem (the Sturm-Liouville problem), and the functions

$$X_k = d_k \sin kx \quad (k = 1, 2, \ldots)$$

for $d_k \neq 0$ are eigenfunctions corresponding to these eigenvalues.

We have found all the eigenvalues and eigenfunctions of the stated Sturm-Liouville problem, since for any $\mu \leqslant 0$ differential equation (7) has only a trivial (equal identically to zero) solution satisfying the conditions $X(0) = X(\pi) = 0$.

Indeed, for $\mu = -\lambda^2 < 0$ the general solution of equation (7) has the form $X = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$. We find the constants C_1 and C_2 from condition (9):

$$\left.\begin{aligned} 0 &= C_1 + C_2, \\ 0 &= C_1 e^{\lambda\pi} + C_2 e^{-\lambda\pi}. \end{aligned}\right\}$$

The determinant of this system

$$\Delta = \begin{vmatrix} 1 & 1 \\ e^{\lambda\pi} & e^{-\lambda\pi} \end{vmatrix} = e^{-\lambda\pi} - e^{\lambda\pi} \neq 0,$$

therefore the system has only a trivial solution $C_1 = C_2 = 0$. Hence, there is no particular solution of equation (7) not equal identically to zero and satisfying conditions (9) for $\mu = -\lambda^2 < 0$.

If $\mu = 0$, then the characteristic equation has the number zero as a multiple root, therefore the general solution of (7) will be written as

$$X(x) = C_1 + C_2 x.$$

Taking into account the boundary conditions, we get $C_1 = 0$, $C_1 + C_2\pi = 0$, whence $C_1 = C_2 = 0$ and $X(x) \equiv 0$.

It remains to solve equation (8) for the found $\mu_k = k^2$:

$$T' + a^2k^2T = 0, \quad \frac{dT}{T} = -a^2k^2\,dt, \quad T(t) = A_k \exp(-a^2k^2t),$$

where A_k is an arbitrary constant.

Thus,

$$u_k(x, t) = b_k \exp(-a^2k^2t) \sin kx \quad (b_k = A_k d_k) \tag{10}$$

are particular solutions of equation (1) satisfying boundary conditions (3).

As is easily seen, the arbitrary finite sum

$$u_N(x, t) = \sum_{k=1}^{N} b_k \exp(-a^2k^2t) \sin kx,$$

where b_k are arbitrary constants, in turn represent a solution of differential equation (1) satisfying the boundary conditions $u_N(0, t) = u_N(\pi, t) = 0$. But then the sum of the infinite series

$$u(x, t) = \sum_{k=1}^{\infty} b_k \exp(-a^2k^2t) \sin kx \tag{11}$$

will also satisfy both equation (1) and boundary conditions $u(0, t) = u(\pi, t) = 0$ for sufficiently small coefficients b_k.

We now select the numbers b_k so that the equality

$$u(x, 0) = f(x) = \sum_{k=1}^{\infty} b_k \sin kx \quad (0 \leqslant x \leqslant \pi) \tag{12}$$

is fulfilled. The numbers b_k are selected in a unique way, namely, by the formula

$$b_k = \frac{2}{\pi} \int_0^{\pi} f(t) \sin kt \, dt,$$

i.e. they must be Fourier coefficients of the function f (see Sec. 4.3).

If the function $f(x)$ is continuous on $[0, \pi]$, then series (12) converges to it in the sense of mean square on $[0, \pi]$ (see Sec. 4.10, Theorem 2).

If it turns out that the series

$$\sum_{k=1}^{\infty} |b_k| < \infty$$

converges, then owing to the inequalities

$$|b_k \exp(-a^2k^2t) \sin kx| \leqslant |b_k|, \tag{13}$$

series (11) converges uniformly and its sum is a continuous function for $t \geqslant 0$.

For $t > 0$ series (11) converges very rapidly—it can be termwise differentiated as many times as desired. In particular,

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= -\sum_{k=1}^{\infty} k^2 b_k \exp(-a^2k^2t) \sin kx, \\ \frac{\partial u}{\partial t} &= -a^2 \sum_{k=1}^{\infty} k^2 b_k \exp(-a^2k^2t) \sin kx, \end{aligned}\right\} \tag{14}$$

whence it is seen that

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

The validity of equalities (14), i.e. of the termwise differentiability of series (11) for $t > 0$ can be traced in the following way. Given $t > 0$, we take t_0 so that $0 < t_0 < t$. Then, for instance, in the case of the first series of (14), assuming that $|b_k| \leqslant M$, we shall have

$$|k^2 b_k \exp(-a^2k^2t) \sin kx| \leqslant Mk^2 \exp(-a^2k^2t_0).$$

But the series of positive (constant!) numbers

$$M \sum_{k=1}^{\infty} k^2 \exp(-a^2k^2t_0) < \infty$$

converges (which can be checked by D'Alembert's or Cauchy's test), and this together with estimate (13) show that the termwise differentiation performed twice with respect to x is legitimate.

Remark. Above we obtained that the Sturm-Liouville problem

$$X'' + \mu X = 0, \quad X(0) = X(\pi) = 0 \tag{15}$$

has the eigenvalues

$$\mu_1 = 1, \quad \mu_2 = 4, \quad \mu_3 = 9, \ \ldots, \quad \mu_k = k^2, \ \ldots. \tag{16}$$

They are positive and to them there correspond the eigenfunctions

$$\sin x, \quad \sin 2x, \quad \sin 3x, \ \ldots, \quad \sin kx, \ \ldots, \tag{17}$$

forming an orthogonal system on the interval $[0, \pi]$:

$$\int_0^{\pi} \sin kx \sin lx \, dx = 0 \quad (k \neq l).$$

From the theory of trigonometric series it is known that system (17) is complete in $L_2'(0, \pi)$ (see Sec. 4.10, Theorem 2), i.e. according to this system, the Fourier series of an arbitrary piecewise continuous on $[0, \pi]$ function converges to this function in the mean square.

For generalization of these facts see Sec. 5.10.

Sec. 5.6. Heat Conductivity for an Infinite Rod

According to Sec. 5.5, the temperature $u(x, y)$ of point x of a rod at time t satisfies the differential equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

We shall consider an infinite rod $(-\infty < x < \infty)$. There are no boundary conditions in this case, therefore we shall seek for a bounded solution of equation (1) satisfying only the initial condition

$$u|_{t=0} = u(x, 0) = \varphi(x), \tag{2}$$

where the function $\varphi(x)$ is defined throughout the entire real axis. We shall assume that the function φ is continuous and belongs to $L'(-\infty, \infty)$. This problem will be called Cauchy's problem for equation (1).

To simplify the notation, we shall set $a = 1$.

To solve the above stated problem, let us apply the Fourier method of separating the variables. We shall seek for a particular solution in the form

$$u(x, t) = X(x) T(t).$$

Substituting this function into (1), we obtain

$$\frac{X''}{X} = \frac{T'}{T} = \mu,$$

$$T' = \mu T, \tag{3}$$

$$X'' = \mu X. \tag{4}$$

The solution of equation (3) has the form

$$T(t) = C \exp(\mu t).$$

From physical considerations it is clear that the temperature $u(x, t) = X(x)\, T(t)$ cannot increase without bound as $t \to \infty$. Hence, the constant μ must be negative. Let us set $\mu = -\lambda^2$. Then the solution of equation (4) will be

$$X(x) = A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x.$$

The function

$$u_\lambda(x, t) = [\alpha(\lambda) \cos \lambda x + \beta(\lambda) \sin \lambda x] \exp(-\lambda^2 t) \qquad (5)$$

is a particular solution of equation (1) for $\forall \lambda$. But then the sum of such solutions and even the integral of function (5) with respect to the parameter λ will also be a solution of equation (1):

$$u(x, t) = \int_0^\infty [\alpha(\lambda) \cos \lambda x + \beta(\lambda) \sin \lambda x] \exp(-t^2 \lambda)\, d\lambda. \qquad (6)$$

Of course, the functions $\alpha(\lambda)$ and $\beta(\lambda)$ must decrease to zero in a sufficiently rapid manner so that the differentiation of (6) with respect to the parameters x and t is legitimate. The functions $\alpha(\lambda)$ and $\beta(\lambda)$ are found from the initial condition

$$u(x, 0) = \int_0^\infty [\alpha(\lambda) \cos \lambda x + \beta(\lambda) \sin \lambda x]\, d\lambda = \varphi(x). \qquad (7)$$

Let us write the expansion of the function φ into a Fourier series (see Sec. 4.13, (1)):

$$\varphi(x) = \int_0^\infty \left\{ \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\xi) \cos \xi\lambda\, d\xi \right) \cos \lambda x + \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\xi) \sin \lambda\xi\, d\xi \right) \sin \lambda x \right\} d\lambda. \qquad (8)$$

Comparing formulas (7) and (8), we see that

$$\alpha(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\xi) \cos \xi\lambda\, d\xi, \quad \beta(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\xi) \sin \xi\lambda\, d\xi. \qquad (9)$$

Substituting these values into (6), we obtain (see Sec. 4.14, (12))

$$u(x,\ t)=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}\varphi(\xi)\cos\lambda(\xi-x)\,d\xi\right]\exp(-\lambda^2 t)\,d\lambda$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(x+\xi)\left[\int_0^{\infty}\exp(-\lambda^2 t)\cos\lambda\xi\,d\lambda\right]d\xi$$

$$=\left(\lambda=\frac{z}{\sqrt{t}}\right)=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(x+\xi)$$

$$\times\left[\int_0^{\infty}\exp(-z^2)\cos\frac{\xi z}{\sqrt{t}}\,\frac{dz}{\sqrt{t}}\right]d\xi$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(x+\xi)\frac{1}{2}\sqrt{\frac{\pi}{t}}\exp\left(-\frac{\xi^2}{4t}\right)d\xi$$

$$=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(\xi+x)\exp\left(-\frac{\xi^2}{4t}\right)d\xi. \quad (10)$$

Thus, the problem (1), (2) has been solved completely.

Remark 1. If the function φ satisfies the conditions mentioned in Remark 1 to Sec. 5.4, then the solution of Cauchy's problem obtained by formula (10) is continuous and bounded together with their derivatives $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial t}$ and is a unique solution in the class of bounded functions.

Sec. 5.7. Small Vibrations of a String

A *string* is defined as a thin thread working in tension but not in bending. If a loose string is crumpled, then it offers no resistance, but if it is stretched, then some tensions appear in it.

Let the ends of a tight string be fixed at points $x = a$ and $x = b$ of the x-axis. We shall assume that the amount of tension appearing in it is equal to the number T and that the density of the string is equal to the number ρ

throughout its length. At the instant of time $t = 0$ our string will be disturbed from the equilibrium condition by drawing it off with a finger and letting it vibrate freely, i.e. the string will perform small oscillations.

The deviation of the string at any of its points having the abscissa x at the instant of time t will be denoted by

$$u = u(x, t) \quad (a \leqslant x \leqslant b, \quad t \geqslant 0).$$

Let us derive the differential equation which will be satisfied by the function u.

Figure 121 depicts the graph of our string at the time t.

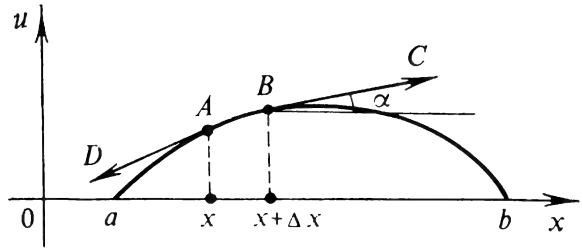

Fig. 121.

Its element, corresponding to the interval $[x, x + \Delta x]$, is acted upon by two tension forces $\overrightarrow{BC}$ and $\overrightarrow{AD}$. The scalar magnitude of either of these forces is equal to T:

$$|\overrightarrow{BC}| = |\overrightarrow{AD}| = T.$$

The force $\overrightarrow{BC}$ is applied to the point B having abscissa $x + \Delta x$, is directed along the tangent to the string at this point and forms with the positive direction of the x-axis an angle α whose tangent is equal to

$$\tan \alpha = \frac{\partial u(x + \Delta x, t)}{\partial x}.$$

Since the string performs small vibrations, we may assume that

$$\tan \alpha \approx \sin \alpha.$$

Hence,

$$\sin \alpha \approx \frac{\partial u(x + \Delta x, t)}{\partial x}.$$

The projection of the force $\overrightarrow{BC}$ on the u-axis is, obviously, equal to

$$T \sin \alpha \approx T \frac{\partial u (x+\Delta x, t)}{\partial x}.$$

And the projection of the force $\overrightarrow{AD}$ on the u-axis is, obviously, equal to

$$-T \frac{\partial u (x, t)}{\partial x}.$$

The sum of these projections is equal to

$$T \frac{\partial u (x+\Delta x, t)}{\partial x} - T \frac{\partial u (x, t)}{\partial x} \approx T \frac{\partial^2 u (x, t)}{\partial x^2} \Delta x.$$

We everywhere neglect the infinitesimals of a higher order than Δx, since we consider the so-called small vibrations of the string.

On the other hand, the product of the mass by the acceleration of the string element under consideration is equal to

$$\rho \Delta x \frac{\partial^2 u (x, t)}{\partial t^2}.$$

Therefore

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} + T \frac{\partial^2 u}{\partial x^2} \Delta x.$$

Reducing by Δx, we obtain the differential equation for the vibrations of the string:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad \left(a^2 = \frac{T}{\rho}\right). \tag{1}$$

Now the mathematical problem in which the study of free vibrations of a string results can be formulated as follows: it is required to solve a linear differential equation (1) with partial derivatives of the second order for the initial conditions

$$u (x, 0) = f (x), \quad \frac{\partial u (x, 0)}{\partial t} = F (x) \tag{2}$$

and for the boundary conditions

$$u (0, t) = u (\pi, t) = 0.$$

Initial conditions (2) show the position occupied by the string at the initial time and what was the velocity of each of its points at $t = 0$. The functions $f(x)$ and $F(x)$ are given functions.

Boundary conditions (3) show that the ends of the string are fixed at the points $a = 0$ and $b = \pi$.

The above formulated problem can be solved by Fourier's method (in the same way as in Sec. 5.5). We first seek the solution of equation (1) in the form of the product

$$u(x, t) = X(x)\, T(t), \tag{4}$$

satisfying the boundary conditions

$$\left.\begin{aligned} X(0)\, T(t) &= 0, \\ X(\pi)\, T(t) &= 0 \end{aligned}\right\} \tag{5}$$

for all $t > 0$. But then

$$X(0) = X(\pi) = 0,$$

since otherwise it would be $T(t) \equiv 0$ and $u(x, t) \equiv 0$. Substituting product (4) into (1), we obtain

$$XT'' = a^2 X'' T,$$

or

$$\frac{1}{a^2}\frac{T''}{T} = \frac{X''}{X}.$$

But a function of t can be equal to a function of x only if both of them are equal to a constant number which will be denoted by $-\mu$:

$$\frac{1}{a^2}\frac{T''}{T} = \frac{X''}{X} = -\mu.$$

As a result, we obtain two ordinary differential equations

$$X'' + \mu X = 0, \quad X(0) = X(\pi) = 0, \tag{6}$$

$$T'' + a^2 \mu T = 0. \tag{7}$$

Equation (6) must be solved with the boundary conditions $X(0) = X(\pi) = 0$, i.e. we have to solve the Sturm-Liouville problem for this equation (see Sec. 5.5). As was shown in Sec. 5.5, the solution of this problem are the numbers (eigenvalues)

$$\mu_k = k^2 \quad (k = 1, 2, \ldots)$$

and the corresponding to them nontrivial functions (eigenfunctions)

$$X_k(x) = \sin kx \quad (k = 1, 2, \ldots),$$

satisfying conditions (5) for these numbers.

The general solution of equation (7) for the found $\mu_k = k^2$ has the form

$$T_k(t) = A_k \cos akt + B_k \sin akt \quad (k = 1, 2, \ldots).$$

Consequently, all the solutions of differential equation (1) of form (4), satisfying boundary conditions (5), can be written in the form

$$u_k(x, t) = (A_k \cos akt + B_k \sin kat) \sin kx$$
$$(k = 1, 2, \ldots),$$

where the constants A_k, B_k for every k may be taken arbitrarily. But then any sums

$$\sum_{k=1}^{N} (A_k \cos akt + B_k \sin akt) \sin kx$$

are also solutions of equation (1) satisfying boundary conditions (5). This property is also possessed by the sums of infinite series

$$u(x, t) = \sum_{k=1}^{\infty} (A_k \cos akt + B_k \sin akt) \sin kx, \qquad (8)$$

if the numbers A_k and B_k tend to zero in a sufficiently rapid manner for these series to be twice differentiated termwise.

Now we have at our disposal a reach supply of functions $u(x, t)$ satisfying equation (1) and boundary conditions (3); they are defined by formula (8), where the numbers A_k, B_k are arbitrary if only the indicated conditions of convergence are fulfilled.

To find the solution of the stated problem, satisfying initial conditions (2), we differentiate (8) with respect to t:

$$\frac{\partial u}{\partial t}(x, t) = \sum_{k=1}^{\infty} ak(-A_k \sin akt + B_k \cos akt) \sin kx \qquad (9)$$

and equate (8) and (9) for $t = 0$ to the given functions $f(x)$ and $F(x)$, respectively:

$$f(x) = \sum_{k=1}^{\infty} A_k \sin kx, \quad F(x) = \sum_{k=1}^{\infty} akB_k \sin kx. \tag{10}$$

Hence,

$$A_k = \frac{2}{\pi} \int_0^{\pi} f(t) \sin kt\, dt, \quad B_k = \frac{2}{\pi ak} \int_0^{\pi} F(t) \sin kt\, dt. \tag{11}$$

If the functions $f(x)$ and $F(x)$ are continuous on $[0, \pi]$, then it is sufficient for the numbers A_k, B_k to be computed by the formulas (11), and series (10) will converge to these functions in any case in the sense of the mean square (see Sec. 4.10). Of course, if the functions $f(x)$ and $F(x)$ are not only continuous, but also have continuous derivatives (the third order is sufficient), then the sum of series (8) will automatically have continuous derivatives of the second order.

Sec. 5.8. Vibrations of an Infinite String. D'Alembert's Formula

If a string is very long, then its ends will exert but little influence on the vibrations occurring somewhere at its middle.

Therefore, when considering free vibrations of an infinite string, we must solve the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

only for the initial conditions

$$u(x, 0) = f(x), \tag{2}$$

$$u'_t(x, 0) = F(x). \tag{3}$$

Such problem is called Caushy's problem or a problem with initial conditions.

This problem is conveniently solved in the following way. Let us introduce the new variables

$$\xi = x - at, \quad \eta = x + at. \tag{4}$$

then equation (1) will turn into the equation

$$\frac{\partial^2 u}{\partial \xi \, \partial \eta} = 0. \tag{5}$$

The solution of equation (5) is, obviously, the function

$$u = \varphi(\xi) + \psi(\eta),$$

where φ and ψ are arbitrary functions which we shall regard as twice differentiable.

Returning to the old variables, we obtain the solution of equation (1) in the form

$$u(x, t) = \varphi(x - at) + \psi(x + at). \tag{6}$$

Direct differentiation of (6) readily shows that this is really so. We have

$$u'_x = \varphi'(x - at) + \psi'(x + at),$$

$$u'_t = -a\varphi'(x - at) + a\psi'(x + at),$$

$$u''_{xx} = \varphi''(x - at) + \psi''(x + at),$$

$$u''_{tt} = a^2\varphi''(x - at) + a^2\psi''(x + at),$$

i.e.

$$a^2 u''_{xx} = u''_{tt}.$$

The obtained solution (6) depending on two arbitrary functions is called *D'Alembert's* solution.*

Using the initial conditions, we find the functions φ and ψ:

$$\varphi(x) + \psi(x) = f(x), \tag{7}$$

$$-a\varphi'(x) + a\psi'(x) = F(x). \tag{8}$$

Integrating (8) over the interval $[0, x]$, we obtain

$$-\varphi(x) + \psi(x) = \frac{1}{a} \int_0^x F(y)\, dy + C, \tag{9}$$

* D'Alembert, Jean le Rond (1717-1783). Outstanding French mathematician, philosopher, and physicist.

where C is an arbitrary constant. From (7) and (9) we find

$$\left.\begin{aligned} \varphi(x) &= \frac{1}{2} f(x) - \frac{1}{2a} \int_0^x F(y)\, dy - \frac{C}{2}, \\ \psi(x) &= \frac{1}{2} f(x) + \frac{1}{2a} \int_0^x F(y)\, dy + \frac{C}{2}. \end{aligned}\right\} \tag{10}$$

Now the solution of Cauchy's problem is written as

$$\begin{aligned} u(x, t) &= \varphi(x - at) + \psi(x + at) \\ &= \frac{1}{2} f(x - at) - \frac{1}{2a} \int_0^{x-at} F(y)\, dy + \frac{1}{2} f(x + at) \\ &\qquad + \frac{1}{2a} \int_0^{x+at} F(y)\, dy, \end{aligned}$$

or

$$u(x, t) = \frac{f(x - at) + f(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} F(y)\, dy. \tag{11}$$

Formula (11) is known as *D'Alembert's formula*.

Sec. 5.9. Vibration of a Circular Membrane

Let a circular membrane of radius 1 occupy at rest a circle of radius 1 centred at the origin of a polar coordinate system (r, φ) (Fig. 122). We shall assume that the membrane is fastened around its periphery ($r = 1$). If now the membrane is acted upon by some force, then the displacement u of points belonging to the membrane will be a function of the coordinates r, φ and time t:

$$u = u(r, \varphi, t).$$

In the same way as in Sec. 5.7, we can obtain the equation for the vibrations of the membrane

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} \right). \tag{1}$$

For equation (1) we shall solve the problem with the boundary condition

$$u\,|_{r=1} = 0 \tag{2}$$

and the initial conditions

$$u\,|_{t=0} = f(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(r). \tag{3}$$

Thus, we consider axially symmetric vibrations of the membrane in which all the points of a circle of radius $r \leqslant 1$

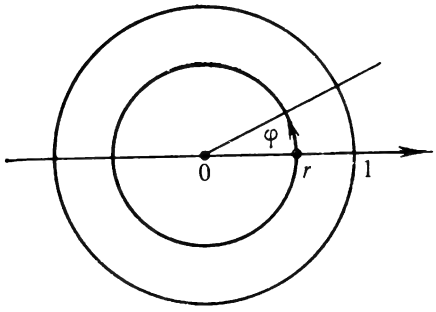

Fig. 122.

centred at the origin have at the initial instant of time velocities and displacements independent of the angle φ. Hence, the function u will also depend only of r and t, and equation (1) is simplified:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right). \tag{4}$$

The solution of equation (4) with conditions (2), (3) can be found by Fourier's method. We first seek for all the solutions of the form

$$u(r, t) = U(r)\, T(t).$$

It can be readily shown (as in Sec. 5.7) that the functions $T(t)$ and $U(r)$ satisfy the equations

$$T'' + \mu^2 a^2 T = 0, \tag{5}$$

$$U'' + \frac{1}{r} U' + \mu^2 U = 0. \tag{6}$$

Thus, we have to find the numbers μ for which equation (6) has a nontrivial solution $U(r)$ satisfying the boundary condition

$$U(1) = 0. \tag{7}$$

These numbers μ are called the eigenvalues of the given Sturm-Liouville problem and the functions $U(r)$, belonging to them, are termed the eigenfunctions.

Remark 1. In Sec. 5.7 we dealt with the Sturm-Liouville problem for a second-order differential equation with two boundary conditions ($X(0) = X(\pi) = 0$). In the present problem we also have a second-order equation, but this time only with one boundary condition ($U(1) = 0$). This is because the given equation has a singularity at the point $r = 0$ due to which this equation has both bounded and unbounded solutions. In fact, here we also seek the eigenfunctions satisfying the two boundary conditions: the first condition consists in that the eigenfunction $U(r)$ must be bounded in the neighbourhood of $r = 0$ and the second condition is $U(1) = 0$.

To obtain the solution of equation (6), let us introduce the new variable

$$\rho = r\mu, \quad V(\rho) = U(\rho/\mu).$$

Then equation (6) turns into a similar equation, but for $\mu = 1$:

$$V''(\rho) + \frac{1}{\rho} V'(\rho) + V(\rho) = 0. \tag{8}$$

Equation (8) was already studied in Sec. 1.24, Example 2. It has two linearly independent solutions: one of them is bounded in the neighbourhood of the point $\rho = 0$, the other being Bessel's function of the zero order

$$J_0(\rho) = \sum_{k=0}^{\infty} (-1)^k \frac{\rho^{2k}}{2^{2k}(k!)^2},$$

where the power series standing on the right-hand side is convergent on the interval $-\infty < \rho < \infty$. All possible solutions of equation (8), bounded in the neighbourhood of the zero point, have the form $CJ_0(\rho)$, where C is an arbitrary constant (see Remark 1 to Sec. 1.24). The correspond-

ing functions

$$CJ_0(r\mu)$$

will be just the required solutions of equation (6) bounded on [0, 1].

It now remains to choose μ so as to fulfill the boundary condition

$$J_0(1\cdot\mu) = 0.$$

We see that the number μ must be a root of the function $J_0(r)$. It is well known that the function $J_0(r)$ has an infinite number of zeros: $0 < \mu_1 < \mu_2 < \ldots$. For instance,

$$\mu_1 = 2.40; \quad \mu_2 = 5.52; \ldots .$$

Thus, the numbers μ_k ($k = 1, 2, \ldots$) are the eigenvalues, and $J_0(\mu_k r)$ are belonging to them eigenfunctions of our boundary problem. We may multiply these functions by arbitrary constants $c_k \neq 0$ to obtain again eigenfunctions

$$c_k J_0(\mu_k r) \quad (k = 1, 2, \ldots),$$

belonging to the numbers μ_k.

For $\mu = \mu_k$ the solution of equation (5) will be written as

$$T_k(t) = a_k \cos \mu_k a t + b_k \sin \mu_k a t.$$

The corresponding solutions of partial differential equation (4), satisfying boundary condition (2), has the form

$$u_k(r, t) = (a_k \cos \mu_k a t + b_k \sin \mu_k a t) J_0(\mu_k r),$$

where a_k, b_k are arbitrary constants. But then the sum of the infinite series

$$u(r, t) = \sum_{k=1}^{\infty} (a_k \cos \mu_k a t + b_k \sin \mu_k a t) J_0(\mu_k r) \qquad (9)$$

is also a solution of equation (4), satisfying boundary condition (2), of course, if the numbers a_k and b_k tend to zero in a sufficiently rapid manner to make it possible to twice differentiate these series term-by-term.

To find the solution of the stated problem, we find the coefficients a_k and b_k from initial conditions (3):

$$u\,|_{t=0} = \sum_{k=1}^{\infty} a_k J_0(\mu_k r) = f(r), \tag{10}$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = \sum_{k=1}^{\infty} a\mu_k b_k J_0(\mu_k r) = F(r). \tag{11}$$

The Bessel functions $J_0(\mu_k r)$ possess the properties similar to those of trigonometric functions. For instance, if the function $f(r)$ is piecewise smooth on [0, 1], then it is certainly expanded in a series of form (10) (with constant coefficients a_k) which converges to it, at least in the mean square (see also Sec. 5.10).

We know that trigonometric functions are orthogonal on $[0, 2\pi]$. The Bessel functions $J_0(\mu_k x)$ are also orthogonal on [0, 1], but, as we say, with respect to the weight function x:

$$\int_0^1 x J_0(\mu_k x)\, J_0(\mu_l x)\, dx = 0 \quad (k \neq l). \tag{12}$$

Hence it follows, as it will be proved below, that in equalities (10), (11) the numbers a_k, b_k are computed necessarily by the formulas

$$\left.\begin{aligned} a_k &= \frac{\int_0^1 x J_0(\mu_k x)\, f(x)\, dx}{\int_0^1 x\,[J_0(\mu_k x)]^2\, dx}, \\ a\mu_k b_k &= \frac{\int_0^1 x J_0(\mu_k x)\, F(x)\, dx}{\int_0^1 x\,[J_0(\mu_k x)]^2\, dx} \\ &(k = 1, 2, \ldots). \end{aligned}\right\} \tag{13}$$

A system of continuous on the interval $[a, b]$ functions

$$\varphi_1(x),\ \varphi_2(x),\ \varphi_3(x),\ \ldots \tag{14}$$

is called normal (orthogonal) on $[a, b]$ with respect to the weight function $\rho(x) \geqslant 0$, where $\rho(x)$ is a continuous function if the equalities

$$\int_a^b \rho(x)\,\varphi_k(x)\,\varphi_l(x)\,dx = 0 \quad (k \neq l) \tag{15}$$

are fulfilled.

If the function $f(x)$ is expanded in a uniformly convergent on $[a, b]$ series in functions $\varphi_k(x)$ of system (14):

$$f(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x), \tag{16}$$

then

$$c_m = \frac{\int_a^b \rho(x)\,\varphi_m(x)\,f(x)\,dx}{\int_a^b \rho(x)\,[\varphi_m(x)]^2\,dx} \quad (m = 1, 2, \ldots). \tag{17}$$

Indeed, on multiplying the series by $\rho(x)\,\varphi_m(x)$, its uniform convergence is not violated and termwise integration over $[a, b]$, by virtue of properties (15), leads to the equality

$$\int_a^b \rho(x)\,\varphi_m(x)\,f(x)\,dx = c_m \int_a^b \rho(x)\,[\varphi_m(x)]^2\,dx \quad (m = 1, 2, \ldots),$$

whence formulas (17) are obtained.

Let us prove (12). The function $J_0(\mu_k x)$ satisfies the equation

$$\frac{d^2}{dx^2} J_0(\mu_k x) + \frac{1}{x}\frac{d}{dx} J_0(\mu_k x) + \mu_k^2 J_0(\mu_k x) = 0 \quad (k = 1, 2, \ldots).$$

Multiplying the equation with the index k by $xJ_0(\mu_n x)$ and the equation with the index n by $xJ_0(\mu_k x)$, and subtracting them, we obtain

$$x\left[J_0(\mu_n x)\frac{d^2}{dx^2}J_0(\mu_k x) - J_0(\mu_k x)\frac{d^2}{dx^2}J_0(\mu_n x)\right] + \left[J_0(\mu_n x)\frac{d}{dx}J_0(\mu_k x) - J_0(\mu_k x)\frac{d}{dx}J_0(\mu_n x)\right] + (\mu_k^2 - \mu_n^2)\,xJ_0(\mu_k x)\,J_0(\mu_n x) = 0.$$

It can be readily verified that this equation can be represented in the form

$$\frac{d}{dx}\left\{x\left[J_0(\mu_n x)\frac{d}{dx}J_0(\mu_k x)-J_0(\mu_k x)\frac{d}{dx}J_0(\mu_n x)\right]\right\}$$
$$+(\mu_k^2-\mu_n^2)\,xJ_0(\mu_k x)\,J_0(\mu_n x)=0.$$

Integrating this equality with respect to x between the limits from 0 to 1, we obtain

$$(\mu_k^2-\mu_n^2)\int_0^1 xJ_0(\mu_k x)\,J_0(\mu_n x)\,dx=0 \quad (k\neq n),$$

since the integral of the first term is equal to zero due to the fact that $J_0(\mu_k)=J_0(\mu_n)=0$.

Sec. 5.10. The General Sturm-Liouville Problem

Consider the second-order differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[\lambda\rho(x)-q(x)]\,y(x)=0, \tag{1}$$

where $p(x)$ is a function continuously differentiable on the interval $[0,\ l]$, and $\rho(x)$ and $q(x)$ are functions continuous on this interval. Here $p(x)>0$, $\rho(x)>0$, $q(x)\geqslant 0$ on $[0,\ l]$ and λ is a constant.

Let us state the (Sturm-Liouville) problem: it is required to find all the numbers λ (eigenvalues) for which there exists a nontrivial twice continuously differentiable solution $y(x)$ of equation (1) (an eigenfunction), satisfying the boundary conditions

$$\left.\begin{aligned}\alpha y(0)+\beta y'(0)&=0,\\ \gamma y(l)+\delta y'(l)&=0,\end{aligned}\right\} \tag{2}$$

where α, β, γ, δ are constant numbers.

It is possible to prove that:

(1) There is a countable set of the eigenvalues of the problem

$$\lambda_1<\lambda_2<\lambda_3<\ldots \quad (\lambda_n\to\infty),$$

to which there correspond the eigenfunctions

$$y_1(x),\ y_2(x),\ y_3(x),\ \ldots.$$

(2) For $q(x) \geqslant 0$ all eigenvalues λ_n are positive.

(3) The eigenfunctions on the interval $[0, l]$ form an orthogonal and normed system with respect to the weight function $\rho(x)$:

$$\int_0^l \rho(x)\, y_m(x)\, y_n(x)\, dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

(4) **Steklov's theorem.** Any function $f(x)$, satisfying boundary conditions (2) and having a continuous first derivative and a piecewise continuous second derivative can be expanded in an absolutely and uniformly convergent series of eigenfunctions $y_n(x)$

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \quad c_n = \int_0^l \rho(x)\, y_n(x)\, f(x)\, dx.$$

Problem 1. Solve the equation of free vibrations of a string in a resistance offering medium

$$\frac{\partial^2 u}{\partial t^2} + 2m \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

for the initial and boundary conditions

$$u(x, 0) = f(x), \quad \frac{\partial u(x, 0)}{\partial t} = F(x), \quad u(0, t) = u(\pi, t) = 0.$$

Hint. Suppose that the friction coefficient m is small $(m < a)$.

Answer.

$$u(x, t) = e^{-mt} \sum_{k=1}^{\infty} (a_k \cos q_k t + b_k \sin q_k t) \sin kx,$$

where

$$q_k = \sqrt{(ka)^2 - m^2},$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx\, dx, \quad b_k = \frac{2}{\pi q_k} \int_0^{\pi} F(x) \sin kx\, dx + \frac{m a_k}{q_k}.$$

Problem 2. Solve the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

for the conditions

$$\frac{\partial u(0, 1)}{\partial x} = 0, \quad u(\pi, t) = u_0, \quad u(x, 0) = \varphi(x).$$

Answer.

$$u(x, t) = u_0 + \sum_{k=1}^{\infty} c_k e^{-a^2\lambda_k^2 t} \cos \lambda_k x,$$

where

$$\lambda_k = \frac{2k+1}{2}, \quad c_k = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \cos \lambda_k x \, dx - \frac{2u_0}{\pi\lambda_k}.$$

Hint. The solution should be sought for in the form $u = u_0 + v(x, t)$, where $v(x, t)$ is an unknown function.

Problem 3. Prove property (3) of the orthogonality of eigenfunction on $[0, l]$ satisfying boundary conditions (2).

Hint. Follow the scheme for proving the orthogonality of the Bessel functions (see Sec. 5.9).

Problem 4. Reduce the (Chebyshev) equation

$$(1 - x^2) y'' - xy' + n^2 y = 0 \tag{3}$$

to form (1) on $[-1, 1]$.

Hint. Multiply both sides of equation (3) by $\rho(x) > 0$ and find the function $\rho(x)$ from the condition

$$-x\rho(x) = [(1 - x^2)\rho(x)]' \quad (\rho(x) = 1/\sqrt{1 - x^2}).$$

Problem 5. Find the weight function $\rho(x)$ for the (Chebyshev-Laguerre) equation

$$xy'' + (1 - x) y' + ny = 0.$$

Answer. $\rho(x) = \exp(-x)$, $0 \leqslant x < \infty$.

Sec. 5.11. Dirichlet's Energy Integral

Let there be given in space (x, y, z) a sufficiently smooth closed contour Γ' defined in the parametric form by the equations

$$x = \varphi(s), \quad y = \psi(s), \quad z = \chi(s) \quad (0 \leqslant s \leqslant l). \tag{1}$$

The projection Γ' on the xy-plane

$$x = \varphi(s), \quad y = \psi(s) \tag{2}$$

will be denoted by Γ. We shall assume that Γ' is projected on the xy-plane in a one-to-one manner, i.e. the contour Γ is non-self-intersecting and bounds some domain Ω of the xy-plane (Fig. 123).

Let a membrane be tightened around Γ', i.e. a film working in tension, but not in bending. It is required to find its equation

$$z = u(x, y), \quad (x, y) \in \Omega. \tag{3}$$

The edge of the membrane is fixed around Γ', therefore the function $u(x, y)$ satisfies the boundary condition

$$u|_{\Gamma} = \chi(s) = u(\varphi(s), \psi(s)), \quad 0 \leqslant s \leqslant l. \tag{4}$$

It is possible to prove that the potential energy of the membrane is expressed, to the accuracy of the factor characterizing its physical properties, by the multiple (double) integral

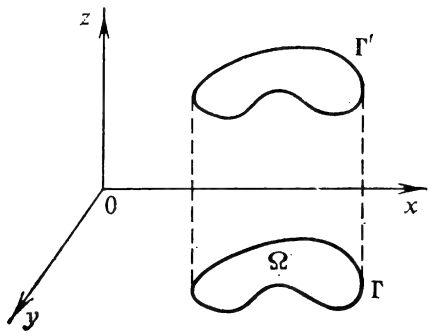

Fig. 123.

$$D[u] = \iint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy, \tag{5}$$

which is called *Dirichlet's integral* of the function $u(x, y)$.

If, with the aid of impressed forces, the membrane is given a new shape

$$z = v(x, y), \quad (x, y) \in \Omega,$$

and is fastened to Γ' as before, then its energy will be equal to

$$D[v] = \iint_{\Omega} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] dx\, dy, \tag{5'}$$

and the membrane itself will continue to satisfy the boundary conditions

$$u\,|_\Gamma = v\,|_\Gamma = \chi(s) = v[\varphi(s), \psi(s)] \quad (0 \leqslant s \leqslant l). \tag{4'}$$

In this state it will be tensioned to a greater extent, therefore $D[v] \geqslant D[u]$.

Consequently, $u(x, y)$ can be defined as such a function whose Dirichlet's integral turns into a minimum among the energy integrals for all possible indicated functions v:

$$D[u] = \min_{v\,|\,\Gamma = \chi(s)} D[v]. \tag{6}$$

Let us introduce the class W of functions $v(x, y)$ having continuous partial derivatives on $\bar{\Omega} = \Omega + \Gamma$ and satisfying the same boundary conditions as u:

$$v\,|_\Gamma = u\,|_\Gamma.$$

There is an infinitude of functions v belonging to the class W. One of them is the desired function $u(x, y)$ reducing $D[v]$ to a minimum among all $v \in W$:

$$D[u] = \min_{v \in W} D[v]. \tag{7}$$

Note that if, by virtue of a certain law, to every function f from some class of functions E ($f \in E$) there corresponds a number $F(f)$, then $F(f)$ is said to be a functional defined on the class of functions E.

Dirichlet's integral $D[v]$ is an example of a functional defined on the class of functions W.

For functionals, the same as for ordinary functions, it is possible to consider the theory of extrema called the calculus of variations. The increment or variation of the argument of the functional $F(f)$ is defined as the difference between two functions from class E:

$$\delta f = f(x) - f_1(x).$$

By analogy with the notion of the differential of a function

$$df = \frac{\partial}{\partial \alpha} f(x + \alpha \Delta x)\,|_{\alpha=0} = f'(x + \alpha \Delta x)\,\Delta x\,|_{\alpha=0} = f'(x)\,\Delta x$$

(where α is an arbitrary real number), we may introduce the notion of the variation of a functional

$$\delta F = \frac{\partial}{\partial \alpha} F(f(x) + \alpha \delta f)\,|_{\alpha=0}.$$

The functional $F(f)$ is said to attain a minimum (maximum) on the function $f_0(x)$ if $F(f) \geqslant F(f)$ $(F(f) \leqslant \leqslant F(f_0))$. The following assertion takes place: if a functional having a variation δF attains a minimum (maximum) on the function $f = f_0$, then

$$\delta F(f_0) = 0.$$

Indeed, with $f_0(x)$ and δf fixed, the functional $F(f_0(x) + \alpha \delta f) = \varphi(\alpha)$ is a function of only one variable α, which for $\alpha = 0$ reaches a minimum (maximum), consequently, $\varphi'(0) = 0$ or $\frac{\partial}{\partial \alpha} F(f_0(x) + \alpha \delta f)\,|_{\alpha=0} = 0$, i.e.

$$\delta F(f_0(x)) = 0. \tag{8}$$

Thus, condition (8) is a necessary condition for an extremum of a functional.

Let us return to Dirichlet's integral and introduce an auxiliary class $\mathfrak{M}_0$ of functions w having continuous partial derivatives in $\bar{\Omega}$ and equal to zero on Γ:

$$w\,|_\Gamma = 0.$$

If a function v has the form

$$v = u + w, \quad w \in \mathfrak{M}_0,$$

then obviously, it belongs to W. Conversely, any function $v \in W$ can be written in the form

$$v = u + (v - u) = u + w, \quad w \in \mathfrak{M}_0,$$

since

$$w\,|_\Gamma = (v - u)\,|_\Gamma = v\,|_\Gamma - u\,|_\Gamma = \chi(s) - \chi(s) = 0.$$

Therefore equality (7) can also be written in the form

$$D[u] = \min_{w\,|_\Gamma = 0} D[u + w]. \tag{9}$$

Problem (9) is called the variation problem, the functions $w \in \mathfrak{M}_0$ being termed variations (i.e. the variations of the argument of the functional $D[v]$).

For the function u the minimum of Dirichlet's integral is attained in the class W: the addition of an arbitrary variation w to u leads the Dirichlet's integral out of the minimum, since it becomes greater.

Suppose we are given an arbitrary function $w \in \mathfrak{M}_0$. Multiplying it by some number λ, we again obtain a function $\lambda w \in \mathfrak{M}_0$, therefore $D[u + \lambda w] \geqslant D[u]$, $\forall \lambda$. But

$$D[u+\lambda w] = \int\int_{\Omega} \left[\left(\frac{\partial (u+\lambda w)}{\partial x} \right)^2 + \left(\frac{\partial (u+\lambda w)}{\partial y} \right)^2 \right] dx\, dy$$

$$= \int\int_{\Omega} \left[\left(\frac{\partial u}{\partial x} + \lambda \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} + \lambda \frac{\partial w}{\partial y} \right)^2 \right] dx\, dy$$

$$= \int\int_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy$$

$$+ 2\lambda \int\int_{\Omega} \left[\frac{\partial u}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial w}{\partial y} \right] dx\, dy$$

$$+ \lambda^2 \int\int_{\Omega} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dx\, dy$$

$$= D[u] + 2\lambda D[u, w] + \lambda^2 D[w], \qquad (10)$$

where we have introduced the notation

$$D[u, w] = \int\int_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial w}{\partial y} \right) dx\, dy.$$

Since $u + \lambda w \in W$, then there takes place the inequality $D[u + \lambda w] \geqslant D[u]$ for any λ's which turns into an equality for $\lambda = 0$. Therefore the function $D[u + \lambda w]$ of λ turns into a minimum for $\lambda = 0$.

But then (see (10) and also (8))

$$\frac{d}{d\lambda} D[u+\lambda w] \,|_{\lambda=0} = \{2D[u, w] + 2\lambda D[w]\} \,|_{\lambda=0}$$

$$= 2D[u, w] = 0,$$

and we have proved that the desired function u satisfies the equation

$$D[u, w] = 0$$

or the equation

$$\iint_{\Omega} \left[\frac{\partial u}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial w}{\partial y} \right] dx\, dy = 0 \tag{11}$$

for all $w \in \mathfrak{M}_0$. Equation (11) is called a variation equation.

To compute the integral

$$\iint_{\Omega} \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}\, dx\, dy,$$

we first perform integration with respect to x with y fixed. Integrating by parts, we have

$$\int_{\sigma_y} \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}\, dx\, dy = \frac{\partial u}{\partial x} w \Big|_A^B + \frac{\partial u}{\partial x} w \Big|_C^D - \int_{\sigma_y} \frac{\partial^2 u}{\partial x^2} w\, dx = -\int_{\sigma_y} \frac{\partial^2 u}{\partial x^2} w\, dx,$$

where σ_y is a section of Ω by a straight line consisting of points having the ordinate y. Depicted in Fig. 124, σ_y consists of two segments $[A, B]$ and $[C, D]$. It should be taken into consideration that the function w is equal to zero at the points A, B, C, D lying on Γ; consequently,

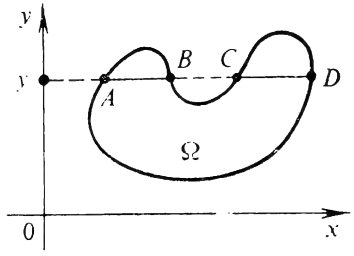

Fig. 124.

$$\iint_{\Omega} \frac{\partial u}{\partial x} \frac{\partial w}{\partial x}\, dx\, dy = -\iint_{\Omega} \frac{\partial^2 u}{\partial x^2} w\, dx\, dy.$$

Analogously, integrating first with respect to y, and then with respect to x, we obtain

$$\iint_{\Omega} \frac{\partial u}{\partial y} \frac{\partial w}{\partial y}\, dy\, dx = -\iint_{\Omega} \frac{\partial^2 u}{\partial y^2} w\, dx\, dy.$$

In such a case it follows from (11) that

$$\iint_{\Omega} w\, \Delta u\, dx\, dy = 0 \tag{12}$$

for all $w \in \mathfrak{M}_0$, where Δ is Laplace's operator. But then

$$\Delta u = 0, \quad \forall (x, y) \in \Omega.$$

Indeed, if we assume that $\Delta u \neq 0$ at least at one point $(x_0, y_0) \in \Omega$, then, by virtue of the continuity of the function Δu, there is a neighbourhood $U(\delta)$ of the point (x_0, y_0) of radius δ, where the function Δu preserves the sign of the number $\Delta u(x_0, y_0)$, the latter being regarded as positive. Then, taking the function

$$w(x, y) = \begin{cases} [(x - x_0)^2 + (y - y_0)^2 - \delta^2]^2, & (x, y) \in U(\delta), \\ 0, & (x, y) \in \Omega \setminus U(\sigma), \end{cases}$$

as a function $w \in \mathfrak{M}_0$, we obtain that

$$\iint_{\Omega} w \Delta u \, dx \, dy = \int_{U} \int_{(\delta)} w \Delta u \, dx \, dy > 0,$$

which contradicts (12).

Thus, we have proved that the function $z = u(x, y)$, describing a membrane, satisfies Laplace's equation in Ω, i.e. u is a harmonic function in Ω. The question of finding $u(x, y)$ has been reduced to Dirichlet's problem: It is required to find the function $u(x, y)$, harmonic in Ω, continuous in $\overline{\Omega}$, and satisfying boundary condition (4).

Note. We assume that it goes without saying that the function u has continuous derivatives of the second order in $\overline{\Omega}$ so that the above computations may be regarded as legitimate.

In fact, this takes place whenever the functions φ, ψ, χ have continuous third derivatives with respect to the parameter s.

CHAPTER 6

THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

Sec. 6.1. The Concept of a Function of a Complex Variable

The concept of a complex number was considered in our book *Higher Mathematics. Differential and Integral Calculus* (Chapter 5) as also polynomials $Q_n(z)$ of a complex variable. Polynomials are the simplest example of functions of a complex variable.

We agreed to represent complex numbers as points in the plane with a rectangular coordinate system specified.

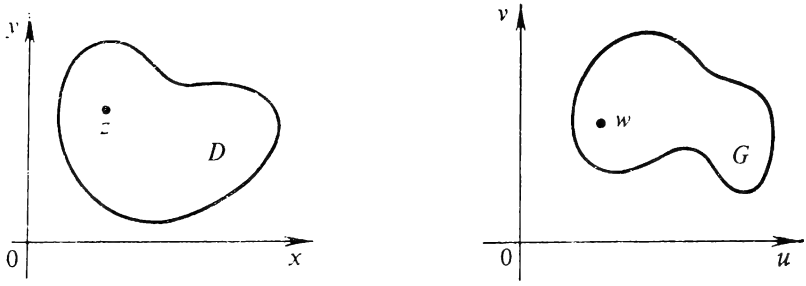

Fig. 125.

Let us introduce the concept of a function of a complex variable.

Let there be given two planes of complex numbers $z = x + iy$ and $w = u + iv$ (Fig. 125). Consider a set of points D in the plane z and a set G in the plane w. If to every number $z \in D$ there corresponds, according to a certain law, a definite complex number $w \in G$, then we say that on the set D a single-valued function of a complex variable is defined which maps the set D into the set G. It is denoted symbolically as:

$$w = f(z).$$

The domain D is called the *domain of definition* of the function $f(z)$. If every point of the set G is a value of the function, then G is said to be the *domain of values* of this function or the *image of the set* D with the aid of the function f ($G = f(D)$). In this case we also say that the function f maps D onto G.

The function $f(z)$ can be written in the form

$$f(z) = u(x, y) + iv(x, y), \quad (x, y) \in D,$$

where

$$\begin{aligned} u(x, y) &= \operatorname{Re} f(z), \\ v(x, y) &= \operatorname{Im} f(z), \end{aligned} \quad (x, y) \in D,$$

are real functions of the variables x, y.

If to every $z \in D$ there correspond several different values of w, then the function $w = f(z)$ is called many-valued.

The notions of the limit and continuity of a function of a complex variable are introduced much in the same way as it is done for a function of a real variable, essentially all that has to be done is to write everywhere the modulus of a complex number instead of the absolute value.

The function

$$w = f(z) = u(x, y) + iv(x, y)$$

is said to have a limit at a point z_0 equal to the number $A = a + ib$ if

$$\lim_{|z - z_0| \to 0} |f(z) - A| = 0. \tag{1}$$

In this case we write

$$\lim_{z \to z_0} f(z) = A.$$

In terms of the functions u and v property (1) is written in the form of an equality

$$\lim_{|z - z_0| \to 0} \sqrt{(u - a)^2 + (v - b)^2} = 0 \tag{2}$$

or, which is the same, in the form of two equalities*

$$\lim_{(x, y) \to (x_0, y_0)} u(x, y) = a, \quad \lim_{(x, y) \to (x_0, y_0)} v(x, y) = b. \tag{3}$$

* For the definition of the limit of a function of two variables see our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 8.3.

For the complex functions $f(z)$ and $g(z)$ the following properties of limits hold:

$$\left.\begin{aligned} &\lim_{z\to z_0}[f(z)\pm g(z)] = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z),\\ &\lim_{z\to z_0}[f(z)\,g(z)] = \lim_{z\to z_0} f(z)\lim_{z\to z_0} g(z),\\ &\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{\lim\limits_{z\to z_0} f(z)}{\lim\limits_{z\to z_0} g(z)} \quad \left(\lim_{z\to z_0} g(z)\neq 0\right). \end{aligned}\right\} \qquad (4)$$

Their proof is completely analogous to the case of real functions of a real variable.

Formulas (4) should be usually understood in the following sense: if the limits standing in their right-hand sides exist, then the limits in their left-hand sides are also existent and the corresponding equality is fulfilled.

The function $w = f(z) = u(x, y) + iv(x, y)$ is said to be continuous at a point z_0, if the following property is fulfilled for it:

$$\lim_{z\to z_0} f(z) = f(z_0). \qquad (5)$$

Thus we say that a function f is continuous at a point z_0 if it is defined in a neighbourhood of z_0 including the point z_0 itself and if equality (5) is fulfilled. Equality (5) is equivalent to two equalities:

$$\lim_{(x,\,y)\to(x_0,\,y_0)} u(x, y) = u(x_0, y_0),$$
$$\lim_{(x,\,y)\to(x_0,\,y_0)} v(x, y) = v(x_0, y_0).$$

Consequently, the continuity of f at the point z_0 is equivalent to the continuity of the functions u and v at the point (x_0, y_0).

It follows from (4) that the sum, the difference, the product, and the quotient of the complex functions $f(z)$ and $g(z)$ continuous at a point z_0 is a continuous function at this point. In the case of the quotient the usual stipulation that the divisor is different from zero (i.e. $g(z_0)\neq 0$) should be made.

Example 1. The function $w = |z| = \sqrt{x^2 + y^2}$ is defined on the entire complex plane. Its values are nonnegative numbers. This function is continuous at all the points of

the complex plane:

$$|| z + \Delta z | - | z || \leqslant | \Delta z | \to 0 \quad (\Delta z \to 0).$$

Example 2.

$$w = \operatorname{Arg} z = \arg z + 2k\pi \quad (k = 0, \pm 1, \ldots). \tag{6}$$

This is a many-valued (infinite-valued) function; $\varphi = \arg z$ being the principal value of the argument ($0 \leqslant \varphi < 2\pi$).

Example 3. $w = z$. This is a continuous function:

$$| z + \Delta z - z | = | \Delta z | \to 0 \quad (\Delta z \to 0).$$

But then the function z^n ($n = 2, 3, \ldots$) is also continuous as the product of a finite number of continuous functions.

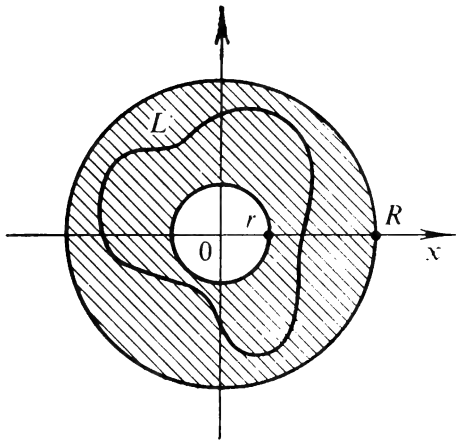

Fig. 126.

A set of complex numbers D will be called a domain if D, as a set of points in the plane, is open and connected.

A domain D is called simply connected if any continuous closed non-self-intersecting curve drawn in D bounds a certain domain G entirely belonging to D. A domain which does not possess this property will be called multiply connected.

Example 4. The annulus $r < | z | < R$ is a multiply (doubly) connected domain. The curve L (Fig. 126) belongs to the annulus but bounds a domain not entirely contained in it.

Sec. 6.2. The Derivative of a Function of a Complex Variable

Let there be given a single-valued function $w = f(z)$ on a domain D (which is an open connected set) of the complex plane z.

The *derivative* $f'(z)$ *of a function* f *at a point* z is defined as

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z) = \frac{dw}{dz}, \qquad (1)$$

when Δz tends to zero in any manner.

By far not every function of a complex variable has a derivative. The existence of limit (1) is a very strong requirement: there must exist the limit indicated in (1) each time $z + \Delta z$ approaches z irrespective of the path traversed.

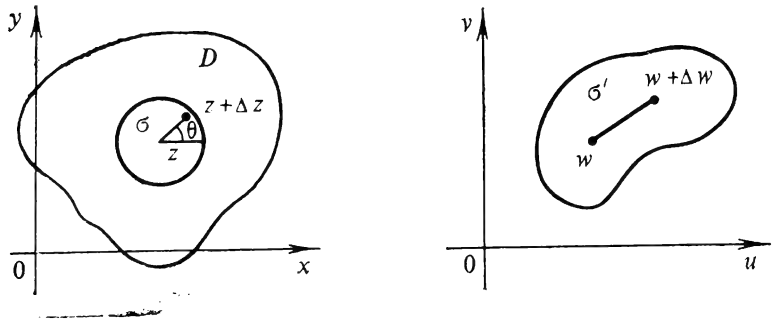

Fig. 127.

A function $f(z)$ having a continuous derivative at any point of a domain D of the complex plane is called an *analytic function on this domain.*

It is possible to prove that *if the derivative of an analytic function* $f(z)$ *is not equal to zero on a domain* D, *then the set* G *of values of the function* $f(z)$ *is also a domain.* We shall use this property below.

Let us give a geometrical representation of the derivative $f'(z)$ when the latter is different from zero. In addition to the plane z, let us introduce another plane of points w. From point z as centre, describe an open circle σ of radius $\delta > 0$ (Fig. 127). An arbitrary point of σ has the form $z + \Delta z$, where Δz is an arbitrary complex number whose modulus is less than δ: $|\Delta z| < \delta$. Let us write Δz in the

exponential form

$$\Delta z = \rho e^{i\theta} \quad (\rho > 0). \tag{2}$$

With the aid of a function $w = f(z)$ the circle σ will go into some domain σ' of the plane w. The domain σ' consists of points $w + \Delta w$, where the increments Δw correspond to all possible indicated increments Δz ($|\Delta z| < \delta$) (see Fig. 127).

From (1) there follows the equality

$$\frac{\Delta w}{\Delta z} = f'(z) + \alpha(\Delta z), \quad \text{where} \quad \alpha(\Delta z) \xrightarrow[\Delta z \to 0]{} 0.$$

Multiplying both sides of the last equality by Δz, we obtain

$$\Delta w = f'(z)\,\Delta z + \Delta z \cdot \alpha(\Delta z). \tag{3}$$

The product $\Delta z \cdot \alpha(\Delta z)$ tends to zero, as $\Delta z \to 0$, faster than Δz. Therefore, if $f'(z) \neq 0$, then the first term of the right-hand member of (3) is the principal one. For sufficiently small Δz we may write approximately, with an accuracy to infinitesimals of a higher order (as compared with Δz)

$$\Delta w \approx f'(z)\,\Delta z.$$

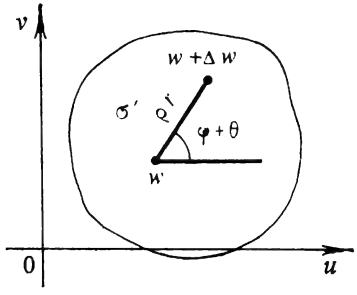

Fig. 128.

Let us write the number $f'(z)$ in the exponential form

$$f'(z) = re^{i\varphi} \quad (r > 0). \tag{4}$$

Therefore, taking into account (2), we obtain

$$\Delta w \approx r\rho e^{i(\varphi+\theta)}$$

$$(|\Delta z| = \rho < \delta).$$

We see that the modulus $|\Delta w|$ is $r = |f'(z)|$ times greater than the modulus $|\Delta z|$ (with an accuracy to an infinitesimal of a higher order):

$$|\Delta w| \approx r\rho = r\,|\Delta z|,$$

and the argument Δw (also to within an infinitesimal of a higher order) is obtained from the argument Δz by adding to it the number φ (Fig. 128):

$$\operatorname{Arg}(\Delta w) \approx \operatorname{Arg}(\Delta z) + \varphi.$$

Thus, in order to imagine where the points $z + \Delta z$ with $|\Delta z| < \delta$ are mapped with the aid of the function $w = f(z)$, it is necessary: (1) to turn the circle σ through an angle of $\varphi = \arg f'(z)$ and (2) to stretch it $r = |f'(z)|$ times. By means of these two operations every point $z + \Delta z$,

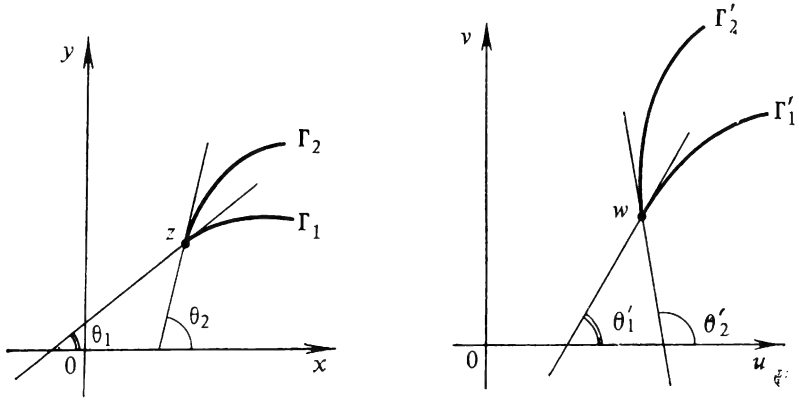

Fig. 129.

$|\Delta z| < \delta$, will go into some point which must then be shifted by the quantity $\Delta z \cdot \alpha$ (Δz) which is an infinitesimal of the order higher than Δz.

Let Γ_1 and Γ_2 be smooth curves emanating from the point z. The tangent lines to them at this point form with the x-axis angles θ_1, θ_2 (measured from the x-axis anticlockwise), respectively. The images of these curves Γ_1', Γ_2' on the plane w (Fig. 129) with the aid of the function $w = f(z)$ have tangent lines at the point w which make with the axis of abscissas angles θ_1', θ_2' (which are also measured anticlockwise). By virtue of property 1,

$$\theta_1' = \theta_1 + \varphi, \quad \theta_2' = \theta_2 + \varphi,$$

whence there follows the property

$$\theta_2' - \theta_1' = \theta_2 - \theta_1,$$

consisting in that the given map preserves angles, also preserving the direction in which the angles are measured (if $\theta_2 > \theta_1$, then $\theta_2' > \theta_1'$).

Besides, as we saw above, the given map realizes at every point z, where $f'(z) \neq 0$, stretching independent of direction.

A map (or mapping) possessing (with an accuracy to infinitesimals of a higher order) the property of preserving angles (with the direction of reference retained) and the property of constant stretchings is called a *conformal map* or *conformal transformation.*

It follows from the aforesaid that the *mapping with the aid of an analytic function* $w = f(x)$ *is conformal at all the points where* $f'(z) \neq 0$.

Remark 1. *If the function* $f(z)$ *of a complex variable* z *has the derivative* $f'(z)$ *everywhere on a domain* D, *then, automatically, this derivative is continuous everywhere on* D, *i.e.* $f(z)$ *is analytic on* D. We shall make use of this statement, although we are not going to prove it.

Remark 2. It follows from equality (3) that if the function $f(z)$ has a derivative at a point z, then it is continuous at this point.

The kth derivative of $f(z)$ is denoted by $f^{(k)}(z)$ and defined by induction

$$(f^{(k-1)}(z))' = f^{(k)}(z) \quad (k = 1, 2, \ldots; f^{(0)}(z) = f(z)).$$

Knowing that the derivative of an analytic function $f(z)$ is continuous on a domain D, we may easily conclude that $f(z)$ has on D continuous derivatives of any order

$$f'(z), f''(z), f'''(z), \ldots .$$

The following terminology is also used: a function $f(z)$ is said to be analytic at a point z_0 if it is analytic in a certain neighbourhood of this point. Finally, the function $f(z)$ is said to be analytic on the closure $\overline{D}$ of a domain D if there exists a domain G containing $\overline{D}$ ($G \supset \overline{D}$), on which $f(z)$ is analytic.

Given below are basic properties of derivatives of functions of a complex variable which are analogous to the corresponding properties of derivatives of functions of a real

variable and therefore are proved in a similar manner:

$$[u(z) \pm v(z)]' = u'(z) \pm v'(z) \tag{5}$$

$$[u(z)\, v(z)]' = u(z)\, v'(z) + u'(z)\, v(z), \tag{6}$$

$$\left[\frac{u(z)}{v(z)}\right]' = \frac{u'(z)\, v(z) - u(z)\, v'(z)}{v^2(z)} \quad (v(z) \neq 0), \tag{7}$$

$$\frac{dw}{dz} = \frac{dw}{dv}\,\frac{dv}{dz}. \tag{8}$$

Formula (8) should be understood in the following way: if w is the function $w = \varphi(v)$ of a complex variable v having the derivative $\frac{dw}{dv} = \varphi'(v)$, and $v = \psi(z)$ is a function of a complex variable z having the derivative $\frac{dv}{dz} = \psi'(z)$, then the derivative of the composite function

$$w = F(z) = \varphi[\psi(z)]$$

is computed by formula (8).

Below, we consider several elementary functions of a complex variable.

The *power function*

$$w = z^n,$$

where n is an integer.

This function has a derivative which is computed by the formula

$$(z^n)' = nz^{n-1} \quad (n = \ldots, -2, -1, 0, 1, \ldots).$$

For $n > 0$ it is conveniently computed as the limit

$$\lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} = nz^{n-1},$$

using the binomial formula.

For $n < 0$ we may now take advantage of formula (7).

For $n \geqslant 0$ the function z^n is analytic on the entire plane z, and for $n < 0$ on the entire plane with the point $z = 0$ deleted.

The *functions* e^z, $\sin z$, $\cos z$, $\tan z$.

The first three of these functions are defined in our book *Higher Mathematics. Differential and Integral Calculus*,

Sec. 9.13, as the sums of power series:

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots,$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

The radius of convergence of each of these series is equal to ∞. Therefore the derivatives of these functions can be obtained for any z by term-by-term differentiation of the corresponding series:

$$(e^z)' = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots = e^z,$$

$$(\sin z)' = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots = \cos z,$$

$$(\cos z)' = -z + \frac{z^3}{3!} - \frac{z^5}{5!} + \ldots = -\sin z.$$

The formulas for the trigonometric functions of the sum of complex arguments remain the same as in the case of a real variable.

The function $\tan z$ is defined by the formula

$$\tan z = \frac{\sin z}{\cos z}.$$

Its derivative

$$(\tan z)' = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} \quad (\cos z \neq 0),$$

which follows from formula (7).

The *function* a^z $(a > 0)$ can be defined by the formula

$$a^z = e^{z \ln a} = \exp(z \ln a).$$

Its derivative is computed on the basis of formula (8) on the derivative of a composite function:

$$(a^z)' = (e^{z \ln a})' = e^{z \ln a} \ln a = a^z \ln a.$$

The *hyperbolic functions* $\sinh z$, $\cosh z$, and $\tanh z$ are defined by the formulas

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \quad \tanh z = \frac{\sinh z}{\cosh z}.$$

Hence it follows that

$$\sinh iz = \frac{e^{iz} - e^{-iz}}{2} = i \sin z, \quad \cosh iz = \cos z, \tag{9}$$

Substituting iz for z in (9), we obtain

$$\cos iz = \cosh z, \quad \sin iz = i \sinh z. \tag{10}$$

Let us also mention the formula

$$\cosh^2 z - \sinh^2 z = 1$$

which is readily verified.

The addition formulas for the hyperbolic functions are easily obtained from (9) and (10) and the corresponding formulas for the trigonometric functions of a complex variable. For instance:

$$\cosh (z_1 + z_2)$$

$$= \cos i (z_1 + z_2) = \cos iz_1 \cos iz_2 - \sin iz_1 \sin iz_2$$

$$= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$$

The derivatives of these functions are computed on the basis of formulas (5), (7), and (8):

$$(\sinh z)' = \left(\frac{e^z - e^{-z}}{2}\right)' = \frac{e^z + e^{-z}}{2} = \cosh z, \quad (\cosh z)' = \sinh z,$$

$$(\tanh z)' = \frac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \frac{1}{\cosh^2 z} \quad (\cosh z \neq 0).$$

Example 1. Separate the real and imaginary parts of the function $w = \cos z$ and find the zero roots of this function.

Let $z = x + iy$, $w = u(x, y) + iv(x, y)$. We have

$$\cos z = \cos (x + iy) = \cos x \cos yi - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y.$$

Thus, $u(x, y) = \cos x \cosh y$, $v(x, y) = -\sin x \sinh y$. To find the zero roots of the function $\cos z$, we must equate its real and imaginary parts to zero:

$$\left.\begin{aligned} \cos x \cosh y &= 0, \\ \sin x \sinh y &= 0. \end{aligned}\right\}$$

Let us solve this system. Since $\cosh y \neq 0$ for any real y, we obtain from the first equation: $\cos x = 0$; for $y \neq 0$ we obtain from the second equation: $\sin x = 0$. For real x's the cosine and sine do not vanish simultaneously, therefore for $y \neq 0$ the system has no solution. But if $y = 0$, then $\sinh y = 0$ and the second equation is satisfied for any x. Thus, the zero roots of the function $\cos z$ are situated on the real x-axis and coincide with those of $\cos x$.

Remark 3. It follows from this assertion that the zero roots of the function $\cosh z$ coincide with those of the function $\cos y$, where $y = \operatorname{Im} z$.

Remark 4. Let us note here that Sec. 6.15 dealing with linear and fractional linear functions may also be read immediately after the present section.

Sec. 6.3. The D'Alembert-Euler (Cauchy-Riemann) Conditions

Consider the complex function

$$w = f(z) = u(x, y) + iv(x, y) \quad (z \in D),$$

defined on a domain D of the complex plane. Let it have a derivative at the point $z \in D$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \tag{1}$$

Thus, irrespective of the way in which $\Delta z = \Delta x + i\,\Delta y$ tends to zero, there must exist limit (1) equal to one and the same complex number $f'(z)$. In particular, it must take place if

(a) $\Delta z = \Delta x + i0 = \Delta x$ and $\Delta x \to 0$,

(b) $\Delta z = 0 + i\,\Delta y = i\,\Delta y$ and $\Delta y \to 0$.

In the first case (see Sec. 6.1, (3))

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

In the second case

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y} + \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y} \right]$$

$$= -i \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

But then there must be fulfilled the equalities

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{2}$$

which are usually called the *Cauchy-Riemann conditions.* Some time Cauchy and Riemann were just supposed to be the first to obtain these conditions. But later on it turned out that the conditions were already known to Euler and D'Alembert.

Thus, we have proved

Theorem 1. *If a function*

$$f(z) = u(x, y) + iv(x, y)$$

has a derivative at a point $z = x + iy$, *then its real components* u *and* v *have at the point* (x, y) *partial derivatives of the first order satisfying the Cauchy-Riemann condition.*

Theorem 1 can be converted, though on the additional assumption that the partial derivatives of u and v are continuous.

Theorem 2. *If the functions* $u(x, y)$ *and* $v(x, y)$ *have at a point* (x, y) *continuous partial derivatives satisfying the Cauchy-Riemann conditions, then the function of a complex variable* $f(z) = u + iv$ *has a derivative at the point* $z = x + iy$.

Proof. Let the functions u and v have continuous partial derivatives at the point (x, y). Then they are differentiable at this point, i.e. their increments corresponding to the

increments Δx, Δy can be written in the form

$$\Delta u = u(x+\Delta x,\ y+\Delta y) - u(x,\ y)$$
$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + o_1(\rho) \quad (\rho \to 0),$$

$$\Delta v = v(x+\Delta x,\ y+\Delta y) - v(x,\ y)$$
$$= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + o_2(\rho) \quad (\rho \to 0),$$

where $\rho = |\Delta z| = \sqrt{\Delta x^2 + \Delta y^2}$, $o_1(\rho)$ and $o_2(\rho)$ $(\rho \to 0)$ are infinitesimal functions of a higher order of smallness than ρ, i.e. $\lim\limits_{\rho \to 0} \frac{o_j(\rho)}{\rho} = 0$ $(j = 1, 2)$. Therefore, taking into consideration that $o_1(\rho) + io_2(\rho) = o(\rho)$ $(p \to 0)$, we have

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$
$$= \frac{\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right)}{\Delta x + i\Delta y} + \frac{o(\rho)}{\Delta z}$$
$$= \frac{\frac{\partial u}{\partial x}(\Delta x + i\Delta y) + \frac{\partial v}{\partial x}(-\Delta y + i\Delta x)}{\Delta x + i\Delta y} + \frac{o(\rho)}{\rho} \cdot \frac{\rho}{\Delta z}$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \underset{\rho \to 0}{o(1)},$$

since $\left|\frac{\rho}{\Delta z}\right| = \frac{\rho}{|\Delta z|} = 1$. The symbol $\underset{\rho \to 0}{o(1)}$ denotes an infinitesimal functions as $\rho \to 0$. Hence,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},$$

i.e. the function f has at the point z a derivative equal to

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{3}$$

Using conditions (2), we can also obtain other forms for expressing the derivative $f'(z)$. The theorem has been proved.

If we take into consideration that the existence of the derivative $f'(z)$ on the domain D automatically implies its continuity on D, then from Theorems 1 and 2 there follows

Theorem 3. *For the function*

$$f(z) = u(x, y) + iv(x, y)$$

to be analytic on a domain D of the plane z, it is necessary and sufficient that the first-order partial derivatives of the functions u and v be continuous on D and the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad ((x, y) \in D)$$

be fulfilled.

The functions u and v are called *conjugate to each other on D.*

Example 1. The functions $|z| = \sqrt{x^2 + y^2}$, $\operatorname{Re} z = x$, $\operatorname{Im} z = y$ are not analytic on the plane z. In fact, each of them can be written in the form $f(z) = u + iv$, where $u \not\equiv 0$ and $v \equiv 0$ are real functions, obviously, not satisfying the Cauchy-Riemann conditions.

Example 2. Check to see whether the Cauchy-Riemann conditions are fulfilled for the real and imaginary parts of the function $w = \cos z$.

In Example 1 of the preceding section we showed that

$$u(x, y) = \cos x \cosh y, \quad v(x, y) = -\sin x \sinh y.$$

Hence,

$$\frac{\partial u}{\partial x} = -\sin x \cosh y, \quad \frac{\partial u}{\partial y} = \cos x \sinh y,$$

$$\frac{\partial v}{\partial x} = -\cos x \sinh y, \quad \frac{\partial v}{\partial y} = -\sin x \cosh y.$$

Thus,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (\forall x, y),$$

i.e. the Cauchy-Riemann conditions are fulfilled.

Since the first-order partial derivatives of the functions u and v are continuous for any points (x, y), the function $w = \cos z$ is analytic throughout the entire complex plane.

Problem 1. Write the functions e^z, $\sin z$, $\sinh z$, $\cosh z$,

z^n (n natural) in the form

$$f(z) = u + iv,$$

where $u = \operatorname{Re} f(z)$, $v = \operatorname{Im} f(z)$ and verify whether they satisfy the Cauchy-Riemann conditions.

Remark 1. If the function $f(z)$ is represented in the form

$$f_1(z) = R(x, y) \exp(i\Phi(x, y)),$$

where R is the modulus and Φ is the argument of the function $f(z)$, then the Cauchy-Riemann conditions have the form

$$\frac{\partial R}{\partial x} = R\frac{\partial \Phi}{\partial y}, \quad \frac{\partial R}{\partial y} = -R\frac{\partial \Phi}{\partial x}.$$

Sec. 6.4. Harmonic Functions

Let on a domain D of the plane z there be given an analytic function $f(z) = u(x, y) + iv(x, y)$. Then, as it was noted in Sec. 6.2, the function $f(z)$ has on D continuous derivatives of any order. But then the functions u and v have on D continuous partial derivatives of any order, and the first derivatives satisfy the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{1}$$

wherefrom it follows:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x\,\partial y}.$$

Adding together these equalities, we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{2}$$

The left-hand member of (2) is denoted by the symbol

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

The equation

$$\Delta u = 0 \tag{3}$$

is called Laplace's equation and the symbol $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y}$ is known as Laplace's operator.

A function u which has continuous partial derivatives of the second order on D and satisfies Laplace's equation (3) is said to be harmonic on D.

Thus, we have ascertained that the real part of a function analytic on D is a harmonic function on D.

Differentiating the first equality of (1) with respect to y and the second with respect to x and subtracting the second equality from the first one, we have

$$\Delta v = 0,$$

i.e. the imaginary part of an analytic function is also a harmonic function.

But the function $f(z) = u + iv$, where u and v are arbitrary functions harmonic on D, is not always analytic on D. It will be analytic only if the functions u and v satisfy the Cauchy-Riemann conditions in this domain.

Let us show that if D is a simply connected domain, then for any function $u(x, y)$ harmonic on D there exists a unique, with the accuracy to an arbitrary constant, conjugate to u on D function v such that

$$f(z) = u(x, y) + iv(x, y)$$

analytic on D.

Let there be given on D a harmonic function $u(x, y)$. We set

$$P(x, y) = -\frac{\partial u}{\partial y}, \quad Q(x, y) = \frac{\partial u}{\partial x}.$$

Since u has in D continuous second-order partial derivatives satisfying Laplace's equation, then

$$\frac{\partial P}{\partial y} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial Q}{\partial x}.$$

From the obtained equality

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on} \quad D$$

and the fact that D is simply connected it follows (see Sec. 3.4) that the line integral

$$\int_{(x_0, y_0)}^{(x, y)} (P\,dx + Q\,dy) = \int_{(x_0, y_0)}^{(x, y)} \left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\right) = v(x, y) \qquad (4)$$

along any piecewise smooth path $L \subset D$ joining the points (x_0, y_0) and (x, y) depends on these points, but is independent of the shape of the path. And v is a function potential for the vector (P, Q) on D, i.e.

$$\frac{\partial v}{\partial x} = P = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = Q = \frac{\partial u}{\partial x}. \tag{5}$$

This shows that v has continuous partial derivatives on D satisfying, together with u, the Cauchy-Riemann conditions. But then u and v are conjugate functions.

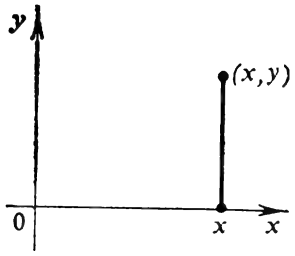

Fig. 130.

If v_1 is another function conjugate to u on D, then

$$\frac{\partial v_1}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v_1}{\partial y} = \frac{\partial u}{\partial x}. \tag{6}$$

It follows from (5) and (6):

$$\frac{\partial (v_1 - v)}{\partial x} = 0, \quad \frac{\partial (v_1 - v)}{\partial y} = 0 \text{ on } D.$$

But then $v_1 - v = C$ on D, where C is a constant. The statement has been proved.

Example 1. The function $u = x^2 - y^2$, obviously, satisfies the equation $\Delta u = 0$. Find the analytic function $f(z)$ in which $\operatorname{Re} f(z) = u$ (Fig. 130).

The imaginary part of this function is found by formula (4):

$$v = \int_{(0,\,0)}^{(x,\,y)} (2y\,dx + 2x\,dy)$$

$$= \int_0^x (2 \cdot 0\,dx + 2x \cdot 0) + \int_0^y (2y \cdot 0 + 2x\,dy) = \int_0^y 2x\,dy = 2xy + C.$$

Then the function $f(z) = (x^2 - y^2) + i(2xy + C) = z^2 + iC$ is analytic throughout the entire complex plane.

Example 2. The function $w = z = x + iy$ is analytic in the z-plane. Consequently, $u = x$, $v = y$ are harmonic functions and satisfy the Cauchy-Riemann conditions in the z-plane. This can be checked directly.

Example 3. The functions $u = x$, $v = -y$ are harmonic ones, but the Cauchy-Riemann conditions are not fulfilled for them, therefore the function $f(z) = x + i(-y) = \bar{z}$ is not analytic.

Let us make sure by direct check:

$$w = \bar{z} = x - iy,$$

$$\Delta w = \overline{(z+\Delta z)} - \bar{z} = \bar{z} + \overline{\Delta z} - \bar{z} = \overline{\Delta z}, \quad \frac{\Delta w}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

We choose two paths along which the point $z + \Delta z$ approaches the point z, namely: (a) $\Delta x = 0$, $\Delta y \to 0$; (b) $\Delta x \to 0$, $\Delta y = 0$. Then:

$$\text{in case (a)} \ \frac{\Delta w}{\Delta z} = \frac{-i\Delta y}{i\Delta y} = -1, \quad \text{i.e.} \ \frac{\Delta w}{\Delta z} \to -1;$$

$$\text{in case (b)} \ \frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta x} = 1, \quad \text{i.e.} \ \frac{\Delta w}{\Delta z} \to 1.$$

Thus, there is no limit $\frac{\Delta w}{\Delta z}$ for $\Delta z \to 0$ and the function $w = \bar{z}$ does not have a derivative at any point of the plane.

Remark 1. In the polar coordinates $x = \rho \cos\theta$, $y = \rho \sin\theta$ the harmonic function $u(x, y)$ will turn into a new function with respect to the coordinates ρ and θ:

$$u(\rho\cos\theta, \ \rho\sin\theta) = \omega(\rho, \ \theta).$$

It is easily seen that

$$\omega'_\rho = u'_x \cos\theta + u'_y \sin\theta,$$

$$\omega''_{\rho^2} = u''_{x^2}\cos^2\theta + 2u''_{xy}\cos\theta\sin\theta + u''_{y^2}\sin^2\theta$$

$$\omega'_\theta = -u'_x\rho\sin\theta + u'_y\rho\cos\theta,$$

$$\omega''_{\theta^2} = u''_{x^2}\rho^2\sin^2\theta - 2u''_{xy}\rho^2\sin\theta\cos\theta + u''_{y^2}\rho^2\cos^2\theta - \rho u'_x\cos\theta - \rho u'_y\sin\theta.$$

Hence,

$$\omega''_{\rho^2} + \frac{1}{\rho}\omega'_\rho + \frac{1}{\rho^2}\omega''_{\theta^2} = u''_{x^2} + u''_{y^2} = \Delta u = 0.$$

In connection with this equality we write

$$\Delta \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$$

and the right-hand side of this symbolic equality is called Laplace's operator in polar coordinates. We have proved that

$$\Delta\omega(\rho, \theta) = 0 \quad \left(\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right).$$

Sec. 6.5. The Inverse of a Function

Let there be given an analytic function

$$w = f(z) \quad (z \in D), \tag{1}$$

mapping a domain D of the z-plane onto a domain G of the w-plane one-to-one. This means that to every $z \in D$ there corresponds, with the aid of function (1), one value $w \in G$ and, by virtue of this law, every $w \in G$ corresponds only to one value $z \in D$. This defines on G a single-valued function

$$z = \varphi(w) \quad (w \in G), \tag{2}$$

possessing the following property:

$$f[\varphi(w)] = w \quad (w \in G).$$

Obviously, there takes place another equality

$$\varphi[f(z)] = z \quad (z \in D).$$

The function $z = \varphi(w)$ is termed the *inverse* of the function $w = f(z)$ $(z \in D)$.

Let us show that if

$$f'(z) \neq 0 \quad (z \in D),$$

then the function $z = \varphi(w)$ is analytic on G.

Indeed, let the points w, $w + \Delta w \in G$. To these points there correspond (with the aid of the inverse function) the points z, $z + \Delta z$. Since, by the hypothesis, the function f has a derivative at the point z, it is continuous at this point: $\Delta w \to 0$ if $\Delta z \to 0$. By virtue of the indicated one-to-one correspondence, the converse is also true (the proof is left to the reader): $\Delta z \to 0$, if $\Delta w \to 0$. But then

$$\lim_{\Delta w \to 0} \frac{\Delta z}{\Delta w} = \lim_{\Delta z \to 0} \frac{1}{\frac{\Delta w}{\Delta z}} = \frac{1}{f'(z)} \quad (f'(z) \neq 0).$$

This shows that the derivative of the inverse function $z = \varphi(w)$ exists at the point w and is equal to

$$\varphi'(w) = \frac{1}{f'(z)} \quad (w \in G). \tag{3}$$

Since w is an arbitrary point in G, $f'(z)$ is a continuous function and $f'(z) \neq 0$, the function $\varphi(w)$ is analytic on G.

Example 1. The function*

$$w = az + b$$

for $a \neq 0$ maps the entire z-plane onto the entire w-plane one-to-one. Here the inverse of the given function has the form

$$z = \frac{w-b}{a}.$$

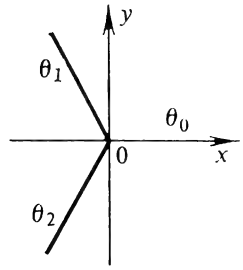

Fig. 131.

It is seen directly that both of these functions are analytic on the planes z and w, respectively $\left(w' = a,\ z' = \frac{1}{a}\right)$.

The *function* $\sqrt[n]{z} = z^{1/n}$ (n natural). Let us cut the plane R of points z into n sectors by the rays

$$\theta = \theta_k = \frac{2\pi}{n} k \quad (k = 0,\ 1,\ \ldots,\ n-1),$$

emanating from the origin (see Fig. 131, where $n = 3$). Let D_k be the sector

$$\theta_k < \theta < \theta_{k+1},\quad \rho > 0, \tag{4}$$

to be more precise, the set of points $z = \rho e^{i\theta}$, $\rho > 0$, having the argument $\theta = \arg z$ satisfying inequalities (4). Obviously, D_k is a domain. Let us also denote by D_k^* the set obtained by adding the ray $\theta = \theta_k$ (together with the origin) to D_k. The points D_k^* can be written in the form

$$z = \rho e^{i\theta} \quad (\theta_k \leqslant \theta < \theta_{k+1},\quad \rho \geqslant 0).$$

Let us also set $\theta = \theta_k + \psi$ $(\theta_k \leqslant \theta < \theta_{k+1})$.

If $0 \leqslant \psi < \theta_1 = 2\pi/n$, then $\theta_k \leqslant \theta < \theta_{k+1}$ and conversely.

---

* For more detail see Sec. 6.15.

The function $w = z^n$ maps D_k^* one-to-one and continuously onto the extended plane

$$w = re^{i\varphi} \quad (0 \leqslant \varphi < 2\pi),$$

which will be denoted by R'.

Indeed,

$$re^{i\varphi} = \rho^n e^{in\theta} = \rho^n e^{in\left(\frac{2\pi}{n}k+\psi\right)} = \rho^n e^{in\psi},$$

therefore

$$r = \rho^n, \quad \varphi = n\psi \quad \left(0 \leqslant \psi < \theta_1 = \frac{2\pi}{n}\right),$$

whence

$$\rho = r^{1/n} = \sqrt[n]{r}, \quad \psi = \varphi/n,$$

where $\sqrt[n]{r}$ is the arithmetic value of the nth root of r, i.e. a non-negative number whose nth power is equal to r. It follows from the aforegoing that the function $w = z^n$ has on the set D_k^* the inverse

$$z = (z)_k = \rho e^{i\theta} = r^{1/n} e^{i\frac{\varphi+2k\pi}{n}},$$
$$k = 0, 1, \ldots, n-1; \quad w \in R'. \tag{5}$$

In general, the function $w = z^n$ has an n-valued inverse function

$$z = \sqrt[n]{w},$$

having n continuous branches (5) corresponding to the numbers $k = 0, 1, \ldots, n-1$. The branches (5) defined by the numbers $k = 0, 1, \ldots, n-1$ map R' onto D_0^*, D_1^*, $\ldots$, $\ldots$, D_{n-1}^*, respectively.

In order to compute the derivative of the kth branch, we have to consider the domain $D_k \subset D_k^*$. Let us denote by R_1' the space R' without the ray $\varphi = 0$.

The analytic function $w = z^n$ maps one-to-one D_k onto R_1', the corresponding inverse being determined by formulas (5). By virtue of (3), its derivative is equal to ($z \in D_k$)

$$\left(\sqrt[n]{w}\right)' = \left(\sqrt[n]{w}\right)_k' = \frac{1}{(z^n)'} = \frac{1}{nz^{n-1}} = \frac{z}{nw}$$

$$= \frac{1}{n} \frac{r^{1/n} e^{i\frac{\varphi+2k\pi}{n}}}{re^{i(\varphi+2k\pi)}} = \frac{1}{n} r^{\frac{1}{n}-1} e^{i\left(\frac{1}{n}-1\right)[\varphi+2k\pi]} = \frac{1}{n} w^{\frac{1}{n}-1}.$$

Considering the domains D_k instead of the sets D_k^*, we exclude the rays $\theta = \theta_k$ of the plane R. If we were interested in the behaviour of the function z^n in the neighbourhood of these rays, then we would have to cut the plane R by the rays

$$\theta = \theta_k + \alpha \quad (0 < \alpha < 2\pi/n, \quad k = 0, 1, \ldots, n - 1)$$

and assume that D_k, D_k^* are the sets of points $z \in R$, defined by the respective inequalities

$$\theta_k + \alpha < \theta < \theta_{k+1} + \alpha, \quad \theta_k + \alpha \leqslant \theta < \theta_{k+1} + \alpha.$$

The *functions* e^z, $\ln z$. The function

$$w = e^z = e^{x+iy} = e^x e^{iy} \quad (z = x + iy),$$

is analytic on the plane R of points z. It is not equal to zero for all $z \in R$. This is seen from the fact that

$$e^x \neq 0 \quad \text{and} \quad |e^{iy}| = 1.$$

We shall denote the plane of points w by R', this plane with the point O removed by R_0', and this plane with the positive ray of the x-axis deleted by R_1'.

We shall see below that the image of R, with the aid of the function $w = e^z$, is the domain R_0'. But the map of R onto R_0' is not one-to-one, since the inverse of the function $w = e^z$ called the natural logarithm of w and denoted as

$$z = \ln w \quad (w \in R_0'),$$

is infinite-valued. Now we are going to define this function.

To this end, R is cut into strips by the straight lines (Fig. 132)

$$y = y_k = 2\pi k$$

$$(k = 0, \pm 1, \pm 2, \ldots).$$

Let us denote the open strip $y_k < y < y_{k+1}$ by D_k and the half-closed strip $y_k \leqslant y < y_{k+1}$ by D_k^*.

The change of the variable y by η with the aid of the equality

$$y = y_k + \eta$$

transforms the strip D_k^* of points $z = x + iy$ into the strip D_1^* of points $x + i\eta$ one-to-one.

Consider the function $w = e^z$ on the set D_k^*. Setting $z = x + iy$, $w = re^{i\varphi}$, $0 \leqslant \varphi < 2\pi$, we have

$$w = re^{i\varphi} = e^x e^{iy} = e^x e^{i(2k\pi+\eta)} = e^x e^{i\eta} \quad (0 \leqslant \eta < 2\pi),$$

whence

$$r = e^x, \quad \varphi = \eta.$$

Thus,

$$x = \ln r = \ln |w|,$$

$$y = y_k + \eta = y_k + \varphi = 2k\pi + \arg w \quad (k = 0, \pm 1, \pm 2, \ldots).$$

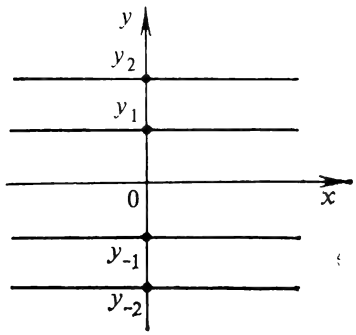

Fig. 132.

Consequently, the function $w = e^z$ has on the strip D_k^* a single-valued inverse function

$$z = x + iy = (z)_k = \ln |w| + i(\arg w + 2k\pi), \quad (6)$$
$$k = 0, \pm 1, \ldots, w \in R_0'.$$

In general, the function $w = e^z$ has an infinite-valued inverse

$$z = \ln w \quad (w \in R_0'),$$

having an infinite number of continuous branches (6) corresponding to the numbers $k = 0, \pm 1, \pm 2, \ldots$.

The domain D_k is transformed by the analytic function $w = e^z$ into the domain R_1' of the w-plane one-to-one. Its single-valued inverse defined for a given k by equality (6) is analytic on R_1'. Its derivative is computed best with the aid of formula (3):

$$(\ln w)' = (\ln w)_k' = \frac{1}{(e^z)'} = \frac{1}{e^z} = \frac{1}{w} \quad (w \in R_1'). \quad (7)$$

Let us underline that here we have computed the derivative not of the many-valued function $\ln w$, but of its definite single-valued branch corresponding to a certain k.

The fact that the derivative turned out to be equal to the function $1/w$, independent of k, is explained by the following: distinct branches (6) differ by a constant.

When computing the derivative of $z = \ln w$, we assumed that points z belong to the domains D_k, the straight lines $y = y_k$ of the plane R being excluded from the consideration.

If we were interested in the behaviour of the considered functions on the straight lines $y = y_k$, then the plane R should be cut by the shifted straight lines

$$y = y_k + \alpha \quad (0 < \alpha < 2\pi,\ y_k = 2\pi k,\ k = 0,\ \pm 1\ \pm 2, \ldots),$$

thus holding that D_k is the domain of points $z = x + iy$, for which $y_k + \alpha < y < y_{k+1} + \alpha$.

The *power function* z^α (α a real number) is defined by the formula

$$w = z = e^{\alpha \ln z} = e^{\alpha[\ln |z| + i(\arg z + 2k\pi)]}$$

$$= |z|^\alpha e^{i\alpha(\arg z + 2k\pi)} \quad (k = 0, \pm 1, \pm 2, \ldots) \qquad (8)$$

or

$$w = \rho^\alpha e^{i\alpha(\theta + 2k\pi)} \quad (k = 0, \pm 1, \pm 2, \ldots), \qquad (8')$$

where

$$z = \rho e^{i\theta}.$$

If α is integral, then

$$e^{i\alpha 2k\pi} = 1$$

and

$$w = (\rho e^{i\theta})^\alpha = z^\alpha,$$

where z^α is understood in the usual sense as the product of α factors z.

If $\alpha = \pm p/q$, where $p > 0$, $q > 0$, are integers, then the numbers on the right in (8′) are essentially different only for $k = 0, 1, \ldots, q - 1$:

$$w = \rho^\alpha e^{i\alpha(\theta + 2k\pi)} \quad (k = 0, 1, \ldots, q - 1).$$

In particular, for $\alpha = 1/n$ (n natural) we already encountered these facts (see (5)).

But if α is an irrational number, then the functions defined by formula (8) or (8′) for different k are different. These are continuous branches of the many-valued (infinite-valued) function $w = z^{\alpha}$.

We further have ($z = \rho e^{i\theta}$)

$$(z^{\alpha})' = (e^{\alpha \ln z})' = e^{\alpha \ln z} \cdot \frac{\alpha}{z} = \alpha \frac{e^{\alpha[\ln \rho + i(\theta + 2k\pi)]}}{e^{[\ln \rho + i(\theta + 2k\pi)]}}$$

$$= \alpha e^{(\alpha - 1)[\ln \rho + i(\theta + 2k\pi)]} = \alpha z^{\alpha - 1},$$

i.e. the equality

$$(z^{\alpha})' = \alpha z^{\alpha - 1}, \tag{9}$$

which is true for any branch z^{α}. Here the following rule should be observed: the branch $z^{\alpha-1}$ on the right-hand side of (9) should be taken with the same k with which the branch z^{α} is taken on the left-hand side of this equality.

Remark 1. The inverse functions for trigonometric and hyperbolic functions can be introduced in a similar way.

For instance, the function $w = \operatorname{Arcsin} z$ is the inverse of the function $z = \sin w$, i.e. $\sin [\operatorname{Arcsin} z] = z$.

From the equation

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{2iw} - 1}{2ie^{iw}}$$

we find

$$e^{2iw} - 2ize^{iw} - 1 = 0, \quad e^{iw} = iz \pm \sqrt{1 - z^2},$$

i. e.

$$iw = \ln \left| iz \pm \sqrt{1 - z^2} \right| + i \left[\arg \left(iz \pm \sqrt{1 - z^2} \right) + 2k\pi \right].$$

Hence,

$$w = \operatorname{Arcsin} z = -i \left[\ln \left| iz \pm \sqrt{1 - z^2} \right| + i \left[\arg \left(iz \pm \sqrt{1 - z^2} + 2k\pi \right) \right] \right]$$

is an infinite-valued function ($k = 0, \pm 1, \pm 2, \ldots$).

Similarly, we can find

$$\operatorname{Arccos} z = -i \left[\ln \left| z \pm \sqrt{z^2 - 1} \right| + i \left[\arg \left(z \pm \sqrt{z^2 - 1} \right) + 2k\pi \right] \right],$$

$$\operatorname{Arctan} z = -\frac{i}{2} \left[\ln \left| \frac{1 + zi}{1 - zi} \right| + i \left(\arg \frac{1 + zi}{1 - zi} + 2k\pi \right) \right],$$

$$\operatorname{Arsinh} z = \ln \left| z \pm \sqrt{z^2 + 1} \right| + i \left[\arg \left(z \pm \sqrt{z^2 + 1} \right) + 2k\pi \right],$$

$$\operatorname{Arcosh} z = \ln \left| z \pm \sqrt{z^2 - 1} \right| + i \left[\arg \left(z \pm \sqrt{z^2 - 1} \right) + 2k\pi \right].$$

Sec. 6.6. Integration of Functions of a Complex Variable

Let $w = f(z) = u + iv$ be a continuous function of a complex z defined in the domain D and L be a smooth curve lying in D with the beginning at the point A and end at the point B (Fig. 133) and defined by the equation

$$z = z(t) = x(t) + iy(t)$$
$$(\alpha \leqslant t \leqslant \beta)$$

or, which is the same, by two equations

$$\left.\begin{aligned} x &= x(t), \\ y &= y(t) \end{aligned}\right\} \quad (\alpha \leqslant t \leqslant \beta). \tag{1}$$

As usual, the direction on L corresponds to the change of the parameter t from α to β ($A = z(\alpha)$, $B = z(\beta)$).

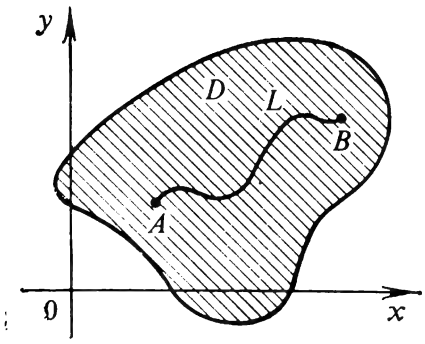

Fig. 133.

The integral of the function $f(z)$ along the curve L is determined in the following way:

$$\int_L f(z)\,dz = \int_L (u + iv)(dx + i\,dy) = \int_L (u\,dx - v\,dy)$$

$$+ i\int_L [v\,dx + u\,dy] = \int_\alpha^\beta [u(x(t), y(t))\,x'(t) - v(x(t), y(t))\,y'(t)]\,dt$$

$$+ i\int_\alpha^\beta [v(x(t), y(t))\,x'(t) + u(x(t), y(t))\,y'(t)]\,dt. \tag{2}$$

Taking into account that $z'(t) = x'(t) + iy'(t)$ and $u(x(t), y(t)) = u(z(t))$, we can write equality (2) in a brief manner:

$$\int f(z)\,dz = \int_{\alpha}^{\beta} f[z(t)]\,z'(t)\,dt. \tag{3}$$

Thus, it is seen from (2) that the integral of f with respect to a complex variable is a sum of two line integrals and that its computation is reduced to computing ordinary integrals.

Integral (2) exists for any continuous function $f(z)$ (in this case the functions $u(x, y)$ and $v(x, y)$ are also continuous) and any smooth curve L (i.e. when $x'(t)$, $y'(t)$ are continuous and $x'(t)^2 + y'(t)^2 > 0$).

If L is a piecewise smooth curve consisting of oriented smooth pieces $L_1, \ldots, L_n$, then, by definition,

$$\int_L f(z)\,dz = \sum_{k=1}^{n} \int_{L_k} f(z)\,dz. \tag{4}$$

On the basis of the properties of a line integral, we easily obtain

(1) $$\int_L f(z)\,dz = -\int_{L^-} f(z)\,dz,$$

where L^- is the same curve, but oppositely oriented (see our book *Higher Mathematics. Differential and Integral Calculus*, Sec. 7.4).

(2) $$\int_L [Af(z) + B\varphi(z)]\,dz = A\int_L f(z)\,dz + B\int_L \varphi(z)\,dz,$$

where A and B are constant numbers.

(3) If $|f(z)| \leqslant M$ for $z \in L$, then

$$\left|\int_L f(z)\,dz\right| \leqslant Ml,$$

where l is the length of L.

Indeed, on the basis of the property of an ordinary integral, we have

$$\left|\int_L f(z)\,dz\right| = \left|\int_\alpha^\beta f[z(t)]\,z'(t)\,dt\right| \leqslant \int_\alpha^\beta |f[z(t)]|\cdot|z'(t)|\,dt$$

$$\leqslant \int_\alpha^\beta M|z'(t)|\,dt = M\int_\alpha^\beta \sqrt{x'(t)^2+y'(t)^2}\,dt = Ml.$$

Example 1.

$$\int_L \frac{dz}{z-z_0} = 2\pi i, \tag{5}$$

where L is a circle with centre at the point z_0 oriented anticlockwise.

Indeed, the equation of L can be written in the form

$$z = z_0 + \rho e^{it} \quad (0 \leqslant t < 2\pi),$$

where ρ is the radius of the circle L. Therefore

$$\int_L \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{\rho i e^{it}\,dt}{\rho e^{it}} = i\int_0^{2\pi} 1\,dt = 2\pi i.$$

Example 2. For integral $n \neq -1$

$$\int_L (z-z_0)^n\,dz = 0, \tag{6}$$

where L is again a circle with centre at the point z_0 oriented anticlockwise.

Indeed,

$$\int_L (z-z_0)^n\,dz = \int_0^{2\pi} \rho^{n+1} i e^{i(n+1)t}\,dt = i\rho^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt$$

$$= i\rho^{n+1}\,\frac{e^{i(n+1)t}}{i(n+1)}\Big|_0^{2\pi} = 0 \quad (n+1 \neq 0),$$

since $e^{i2(n+1)\pi} = 1$ for any integral n.

Theorem 1. (Cauchy's integral theorem). *If the function $f(z)$ is analytic in a finite simply connected domain D of the complex plane, then the integral of $f(z)$ about any piecewise*

smooth closed curve Γ *in* D *is equal to zero*:

$$\int_{\Gamma} f(z)\,dz = 0.$$

Proof. Since $f(z) = u + iv$ is a function analytic in D, then the functions $u(x, y)$ and $v(x, y)$ are continuously differentiable and the Cauchy-Riemann conditions are fulfilled:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{7}$$

by virtue of which the expressions $v\,dx + u\,dy$ and $u\,dx - v\,dy$ are total differentials of certain functions. Therefore the line integrals of these expressions over the closed curve Γ are equal to zero (see Secs. 3.4 and 3.5). But then, according to equality (2),

$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma} (u\,dx - v\,dy) + i\int_{\Gamma} (v\,dx + u\,dy) = 0.$$

Example 3.

$$\int_{\Gamma} z^n\,dz = 0 \quad (n = 0, 1, 2, \ldots),$$

$$\int_{\Gamma} e^z\,dz = 0, \quad \int_{\Gamma} a^z\,dz = 0 \quad (a > 0),$$

$$\int_{\Gamma} \sin z\,dz = 0, \quad \int_{\Gamma} \cos z\,dz = 0,$$

$$\int_{\Gamma} \sinh z\,dz = 0, \quad \int_{\Gamma} \cosh z\,dz = 0,$$

where Γ is an arbitrary closed piecewise smooth contour, since the integrands are analytic in the z-plane. In fact, they have a continuous derivative at all points z of the complex plane.

As a consequence from Theorem 1, we obtain the following theorem.

Theorem 2. *Let a domain* D *of the complex plane be bounded by a composite positive oriented piecewise smooth contour* Γ, *i.e. when we traverse along* Γ, *the points contained in* D *remain*

at the left. Then for a function $f(z)$, *analytic in* $\overline{D}$, *there takes place the equality*

$$\int_{\Gamma} f(z)\,dz = 0.$$

Let us explain this theorem. Figure 134 represents a doubly connected domain D with a piecewise smooth contour $\Gamma = \Gamma_1 + \Gamma_2$ oriented in the positive direction.

Connect the contours Γ_1 and Γ_2 with a smooth piece γ, as in Fig. 135, and orient γ in two opposite ways γ_+, γ_-.

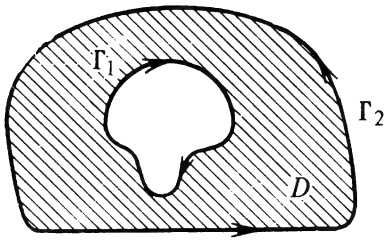

Fig. 134.

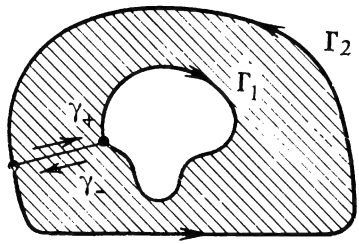

Fig. 135.

As a result, we get a new domain D^1 which is simply connected and is bounded by the oriented contour $\Gamma_2 + \gamma_+ + \Gamma_1 + \gamma_-$. By Theorem 1,

$$\int_{\Gamma_2+\gamma_++\Gamma_1+\gamma_-} f(z)\,dz = 0.$$

But

$$\int_{\gamma_-+\gamma_+} f(z)\,dz = \int_{\gamma_-} f(z)\,dz + \int_{\gamma_+} f(z)\,dz = 0,$$

therefore

$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0.$$

where either of the integrals $\int_{\Gamma_1}$, $\int_{\Gamma_2}$ may be nonzero.

Remark 1. For the sake of brevity, we shall write a "contour" instead of a "closed continuous piecewise smooth contour".

As a consequence of Theorem 2, we get the following theorem.

Theorem 3. *Let a domain D be bounded by an outside contour Γ oriented anticlockwise and inside contours Γ_1,*

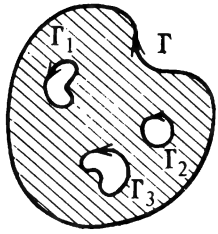

Fig. 136.

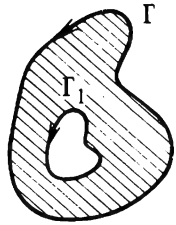

Fig. 137.

$\Gamma_2, \ldots, \Gamma_N$ *oriented also anticlockwise* (as in Fig. 136, where $N = 3$), *and let an analytic function $f(z)$ be given in $\bar{D}$.*

Then the following equality takes place:

$$\int_{\Gamma} f(z)\,dz = \sum_{k=1}^{N} \int_{\Gamma_k} f(z)\,dz. \tag{8}$$

Indeed, if we assume that Γ_k^- is the same contour as Γ_k but oriented clockwise, then, by Theorem 2.

$$\int_{\Gamma} f(z)\,dz + \sum_{k=1}^{N} \int_{\Gamma_k^-} f(z)\,dz = 0,$$

whence follows (8), since

$$\int_{\Gamma_k^-} f(z)\,dz = -\int_{\Gamma_k} f(z)\,dz.$$

Note that if in Theorem 3 $N = 1$, then

$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz \tag{9}$$

(Fig. 137).

Remark 2. From equality (9), i.e. from Theorem 3 for $N = 1$, it follows that equalities (5) and (6) remain true if

in them the circle L centred at the point z_0 is replaced by any closed piecewise smooth contour L' which contains the point z_0 and is oriented anticlockwise:

$$\int_{L'} \frac{dz}{z-z_0} = 2\pi i, \tag{10}$$

$$\int_{L'} (z-z_0)^n \, dz = 0 \quad (n \neq -1). \tag{11}$$

Formulas (10) and (11) are basic in this theory. To them, as we shall see below, the computation of line integrals of analytic functions is reduced (see Secs. 6.10 and 6.11).

Sec. 6.7. Cauchy's Integral Formula

Let $f(z)$ be an analytic function of the complex variable z in a closed simply connected domain $\bar{D}$ ($\bar{D} = D \cup \partial D$)

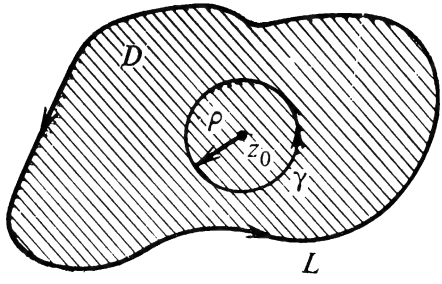

Fig. 138.

with a piecewise smooth boundary L. Then Cauchy's integral formula takes place

$$f(z_0) = \frac{1}{2\pi i} \int_L \frac{f(z)\,dz}{z-z_0}, \tag{1}$$

where z_0 is any point inside the contour L and integration is carried out in the positive direction, i.e. anticlockwise (Fig. 138).

Thus, it is sufficient to define an analytic function on the contour L, and, by formula (1), we can automatically obtain its values at other points of D.

To prove formula (1), let us consider the function

$$\varphi(z)=\frac{f(z)-f(z_0)}{z-z_0}. \tag{2}$$

The function $\varphi(z)$ is analytic at all points of $\bar{D}$ except for $z=z_0$. Let us describe a circle $\gamma \subset D$ about the point z_0 (see Fig. 138). Then, by Theorem 3 from the preceding section,

$$\int_L \varphi(z)\,dz=\int_\gamma \varphi(z)\,dz \tag{3}$$

and the value of the integral

$$\int_\gamma \varphi(z)\,dz=\int_{|z-z_0|=\rho} \varphi(z)\,dz,$$

in fact, is independent of ρ. Note that from (2) it follows that

$$\lim_{z\to z_0}\varphi(z)=f'(z_0).$$

If we define the function $\varphi(z)$ at the point z_0 by setting $\varphi(z_0)=f'(z_0)$, then it becomes continuous in a closed domain $\bar{D}$ and, consequently, its modulus is bounded: $|\varphi(z)|\leqslant M$, $\forall\, z\in\bar{D}$. Hence (see property (3) in Sec. 6.6)

$$\left|\int_\gamma \varphi(z)\,dz\right|\leqslant M\cdot 2\pi\rho.$$

Since the number ρ can be taken arbitrarily small and the integral $\int_\gamma \varphi(z)\,dz$ is independent of ρ, then

$$\int_\gamma \varphi(z)\,dz=0.$$

Therefore from (3) we have

$$\int_L \varphi(z)\,dz=\int_L \frac{f(z)-f(z_0)}{z-z_0}\,dz=0.$$

Since (see (10) in the preceding section)

$$\int_L \frac{f(z_0)}{z-z_0}\,dz = f(z_0)\int_L \frac{dz}{z-z_0} = 2\pi i f(z_0),$$

Cauchy's integral formula has been proved.

Cauchy's formula takes place for a multiply connected domain as well; its proof can be reduced to the above formula proved for a simply connected domain.

Figure 139 represents a doubly connected domain D with a positive oriented boundary L consisting of two closed appropriately oriented contours ($L = L_0 + L_1$).

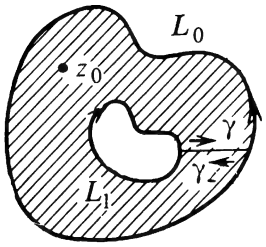

Fig. 139.

Let z_0 be an arbitrary point in D. Let us now connect the contours L_0 and L_1 with a piecewise smooth curve γ oriented from L_1 to L_0 and not passing through the point z_0. Along with the curve γ, we introduce a curve γ_- coinciding with γ but oriented in the opposite direction.

With γ removed from D, the remaining domain D_* will be simply connected with a positive oriented boundary:

$$L' = L_0 + \gamma_- + L_1 + \gamma = L + \gamma_- + \gamma.$$

The function $f(z)$ is analytic on $\bar{D}_*$ and $z_0 \in D_*$. Therefore, by Cauchy's theorem for a simply connected domain,

$$f(z_0) = \frac{1}{2\pi i}\int_{L'} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\left(\int_L + \int_\gamma + \int_{\gamma^-}\right)\frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\int_L \frac{f(z)}{z-z_0}\,dz,$$

since

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz + \int_{\gamma_-} \frac{f(z)}{z-z_0}\,dz = 0.$$

Example 1. Compute the integral

$$\int_L \frac{\sin z}{z^2+1}\,dz,$$

where L is an anticlockwise oriented contour containing the point $z = i$ (Fig. 140) and such that the point $z = -i$ is found outside it.

Let us write our integral in the form

$$\int_L \frac{\sin z\, dz}{(z+i)(z-i)}$$

and consider the function $f(z) = \sin z/(z + i)$. By virtue of our assumptions concerning the contour L, this function is

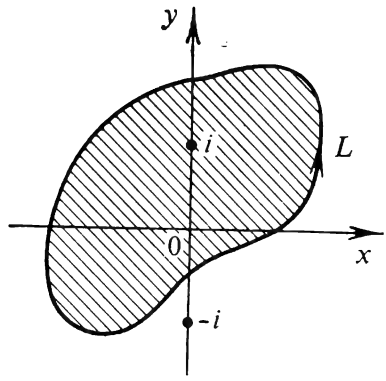

Fig. 140.

analytic in a closed domain bounded by the contour L, therefore by Cauchy's formula

$$\int_L \frac{\sin z\, dz}{z^2+1} = \int_L \frac{f(z)}{z-i}\, dz = 2\pi i f(i)$$

$$= 2\pi i \frac{\sin i}{2i} = \pi \sin i = \pi i \sinh 1.$$

Sec. 6.8. Integral of the Cauchy Type

The expression

$$\frac{1}{2\pi i} \int_L \frac{f(z)\, dz}{z - z_0},$$

where $f(z)$ is an analytic function in a closed domain $\bar{D}$ bounded by a positive oriented contour L, is called *Cauchy's integral.*

If z_0 lies inside L, then the integral equals $f(z_0)$, and if z_0 lies outside L, then $\frac{f(z)}{(z-z_0)}$ is an analytic function in $\bar{D}$ and, consequently, Cauchy's integral is equal to zero.

Let now $\mathscr{L}$ be any piecewise smooth oriented curve, not necessarily closed, and $\varphi(z)$ be a continuous function defined along $\mathscr{L}$. The expression

$$F(z_0) = \frac{1}{2\pi i} \int_{\mathscr{L}} \frac{\varphi(z)}{z - z_0}\, dz \tag{1}$$

is called the *integral of the Cauchy type.* It represents the function $F(z_0)$ defined outside $\mathscr{L}$ ($z_0 \bar{\in} \mathscr{L}$).

Theorem 1. *Integral* (1) *of the Cauchy type is an analytic function* $F(z_0)$ *for all* $z_0 \bar{\in} \mathscr{L}$.

The derivative of order n of $F(z_0)$ *is computed by the formula*

$$F^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\mathscr{L}} \frac{\varphi(z)\, dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, \ldots). \tag{2}$$

Proof. Let σ be an arbitrary circle having no points in common with the curve $\mathscr{L}$. The function of two complex variables z_0 and z

$$\Phi(z_0, z) = \frac{\varphi(z)}{z - z_0}$$

is continuous on the set $\sigma \times \mathscr{L}$ ($z_0 \in \sigma$, $z \in \mathscr{L}$) and has on it a continuous partial derivative

$$\frac{\partial \Phi}{\partial z_0} = \frac{\varphi(z)}{(z - z_0)^2}$$

(it should be taken into consideration that, since the circle σ does not intersect $\mathscr{L}$, for any $z_0 \in \sigma$ and $z \in \mathscr{L}$ the difference $z - z_0 \neq 0$). This shows that the differentiation of $F(z_0)$ with respect to the parameter z_0 is legitimate to be performed under the integral sign in (1):

$$F'(z_0) = \frac{1}{2\pi i} \int_{\mathscr{L}} \frac{\varphi(z)\, dz}{(z - z_0)^2}.$$

Here the derivative $F'(z_0)$ is continuous outside $\mathscr{L}$ (see Theorem 4 in Sec. 2.4 which is readily generalized for the case of an integral of a complex variable).

We have proved formula (2) for the case $n = 1$. For $n \geqslant 2$ it is proved by induction.

Corollary 1. *If a function $w = f(z)$ is analytic in a domain D, i.e. has a continuous first derivative in D, then it has derivatives of all orders.*

Proof. Let z_0 be any point of D and σ a circle centred at z_0 lying entirely in the domain D, and γ a circle serving as the boundary of σ oriented anticlockwise. Then, by Cauchy's formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{z - z_0},$$

i.e. the function $f(z_0)$ is represented by an integral of the Cauchy type for $\mathscr{L} = \gamma$ and $\varphi(z) = f(z)$. Hence, by virtue of Theorem 1, $f(z)$ is infinitely differentiable and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 1, 2, \ldots). \tag{3}$$

Sec. 6.9. Power Series

Consider the power series

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \quad (|z - z_0| < R), \tag{1}$$

having the radius of convergence $R > 0$.

As is known from the theory of power series, series (1) is uniformly convergent in a circle $|z| \leqslant \rho$, where ρ is any positive number less than R ($\rho < R$). Therefore the sum $f(z)$ of series (1) is a continuous function in an open circle $|z - z_0| < R$. Moreover, $f(z)$ has in this circle a continuous derivative $f^{(n)}(z)$ of any order which can be computed by term-by-term differentiation of series (1). This shows that the sum of a power series is an analytic function in the circle (open!) of its convergence. The numbers c_k are computed by the formula

$$c_k = \frac{f^{(k)}(z_0)}{k!} \quad (k = 0, 1, 2, \ldots), \tag{2}$$

which shows that a power series is Taylor's series of its sum. By virtue of equalities (3) derived in the preceding

section, this formula may be replaced by the following:

$$c_k = \frac{1}{2\pi i} \int_L \frac{f(z)\,dz}{(z-z_0)^{k+1}} \quad (k = 0,\ 1,\ 2,\ \ldots),$$

where L is an arbitrary anticlockwise oriented contour belonging to the circle of convergence of series (1) and containing an interior point z_0.

The following theorem is also true.

Theorem 1. *A function $f(z)$, analytic in a circle $|z - z_0| < R$, is expanded in a power series in powers of $(z - z_0)$ convergent to this function.*

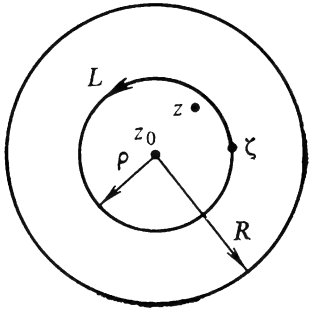

Fig. 141.

Proof. Let $f(z)$ be analytic in a circle $|z - z_0| < R$, and let us denote by z any point lying inside this circle (see Fig. 141). Describe a circle L of radius $\rho < R$ with centre at the point z_0 so that the point z turns out inside the contour L. Then the function $f(z)$ will be analytic on the contour L and inside it. Therefore, by Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i} \int_L \frac{f(\zeta)\,d\zeta}{(\zeta - z)}. \tag{3}$$

The fraction $1/(\zeta - z)$ can be represented in the form

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}}. \tag{4}$$

If the point $\zeta \in L$, and z is found inside this contour, then

$$\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{\rho} < 1. \tag{5}$$

Therefore $1\Big/\left(1 - \dfrac{z - z_0}{\zeta - z_0}\right)$ may be regarded as the sum of a convergent geometric progression

$$\frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = 1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \ldots. \tag{6}$$

From (4) and (6) we obtain

$$\frac{1}{\zeta-z}=\frac{1}{\zeta-z_0}+\frac{z-z_0}{(\zeta-z_0)^2}+\frac{(z-z_0)^2}{(\zeta-z_0)^3}+\ldots, \qquad (7)$$

and series (7) converges uniformly for any $\zeta \in L$ and constant z, since, as is seen from (5), the expression $\left|\frac{z-z_0}{\zeta-z_0}\right|$ is independent of $\zeta \in L$ and is less than 1.

Multiplying (7) by $\frac{f(\zeta)}{2\pi i}$ (without violating its uniform convergence) and integrating along L, we have

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)\,d\zeta}{\zeta-z}=\frac{1}{2\pi i}\int_L \frac{f(\zeta)\,d\zeta}{\zeta-z_0}+\frac{z-z_0}{2\pi i}\int_L \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^2}+\ldots .$$

By virtue of (3),

$$f(z)=c_0+c_1(z-z_0)+c_2(z-z_0)^2+\ldots, \qquad (8)$$

where

$$c_n=\frac{1}{2\pi i}\int_L \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}=\frac{f^{(n)}(z_0)}{n!} \quad (n=0,\ 1,\ 2,\ \ldots). \qquad (9)$$

Thus, we have proved that a function $f(z)$ analytic in a circle $|z-z_0|<R$ is represented by power series (8) with coefficients (9), i.e. by its Taylor series.

Example 1. When expanding functions into Taylor's series, use may be made of the known expansions of elementary functions. For instance,

$$\cos z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!},$$

therefore

$$\sin^2 z=\frac{1-\cos 2z}{2}=\frac{-1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2z)^{2n}}{(2n)!}.$$

Example 2. In a sufficiently small neighbourhood of $z=0$ the function $\tan z=\frac{\sin z}{\cos z}$ is an analytic function ($(\tan z)'=\frac{1}{\cos^2 z}$, $\cos z\neq 0$). Therefore the given function can be expanded into Taylor's series in powers of z, although the

general form of the coefficient is difficult to compute. We have $c_0 = 0$, $c_1 = 1$, $c_2 = 0$, $c_3 = 2$, i. e.

$$\tan z = z + \frac{2z^3}{3!} + \ldots .$$

Example 3. Expand either of the functions $w = \sinh z$ and $w = \cosh z$ into Taylor's series. We have $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, therefore

$$\sinh z = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad \cosh z = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

Sec. 6.10. The Laurent Series *

Theorem 1. *Let* $0 \leqslant r < R \leqslant \infty$. *Any function, analytic in the annulus*

$$r < |z - z_0| < R \tag{1}$$

is uniquely represented in that annulus by the convergent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{c_{-n}}{(z - z_0)^n}, \tag{2}$$

where

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}} \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{3}$$

and γ *is any circle* $|\zeta - z_0| = \rho$, $r < \rho < R$, *oriented anticlockwise.*

---

* Laurent, Paul Mathieu Hermann (1841-1908). French analyst. Best known for Laurent series, which generalizes Taylor's series.

Series (1) is called the *Laurent series of the function* $f(z)$ in powers of $(z - z_0)$ or the *Laurent expansion of the function* $f(z)$ *in the annulus* $r < |z - z_0| < R$.

Remark 1. By saying that the series $\sum_{-\infty}^{\infty}$ is convergent, we mean that the series $\sum_{n=0}^{\infty}$ and $\sum_{n=-\infty}^{-1}$ are separately convergent.

The proof of Theorem 1. Let us take two anticlockwise oriented circles c and C of radii r' and R' centred at the point z_0, where $r < r' < R' < R$ (Fig. 142).

By the hypothesis, $f(z)$ is analytic in the annulus between the circles c and C and on the circles themselves. Therefore, by Cauchy's formula for a composite contour, we have

$$f(z) = \frac{1}{2\pi i}\int_G \frac{f(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{c_-} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

or

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_c \frac{f(\zeta)\,d\zeta}{\zeta - z}, \qquad (4)$$

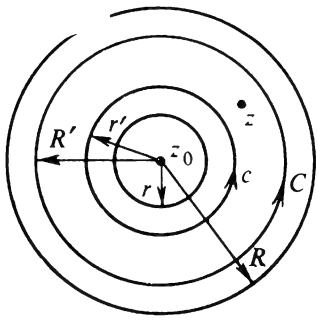

Fig. 142.

where z is a point found between the circles c and C.

In the first integral ζ denotes a point on the circle C, therefore

$$\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{R'} < 1,$$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0)\left[1 - \dfrac{z - z_0}{\zeta - z_0}\right]} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}, \qquad (5)$$

the series on the right being uniformly convergent for $\zeta \in C$ (with z fixed).

In the second integral ζ denotes a point of the circle c, therefore

$$\left|\frac{\zeta-z_0}{z-z_0}\right|=\frac{r'}{|z-z_0|}<1,$$

$$\frac{1}{\zeta-z}=\frac{-1}{(z-z_0)\left[1-\dfrac{\zeta-z_0}{z-z_0}\right]}=-\sum_{n=0}^{\infty}\frac{(\zeta-z_0)^n}{(z-z_0)^{n+1}}, \qquad (6)$$

the series on the right being uniformly convergent for all $\zeta \in C$ (with z fixed).

Substituting (5) and (6) into (4) and performing term-by-term integration, we obtain

$$\begin{aligned} f(z)=&\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_C\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}(z-z_0)^n \\ &+\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_c\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{-n}}(z-z_0)^{-n-1} \\ =&\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_C\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}(z-z_0)^n \\ &+\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_c\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{-n+1}}(z-z_0)^{-n}. \qquad (7) \end{aligned}$$

Since the function $f(\zeta)/(\zeta-z_0)^{n+1}$ is analytic in the annulus for any n, by virtue of Cauchy's theorem, integral (3) is equal to the like integral along any other circle, c and C in particular. Therefore, from (7) follows (2), where the numbers c_n are computed by formulas (3).

The first series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ in the right-hand side of (2) converges in the circle $|z-z_0|<R$, to some function $f_1(z)$, which is analytic in that circle. It is called the *true part of the Laurent series.*

The second series in the right-hand side of (2)

$$\sum_{n=1}^{\infty} c_{-n}(z-z_0)^{-n},$$

converges for $|z - z_0| > r$. It defines an analytic function $f_2(z)$ called the *principal part of the Laurent series.*

Hence,

$$f(z) = f_1(z) + f_2(z),$$

where $f_1(z)$ is a function analytic inside the circle $|z - z_0| < R$ and $f_2(z)$ outside the circle of radius r centred at the point z_0 ($|z - z_0| > r$). Inside the annulus $r < |z - z_0| < R$ both of these functions are analytic.

The coefficients c_n of the Laurent series for the function $f(z)$ are unique, since they are computed by formulas (3).

Example 1. The function

$$f(z) = \frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2}$$

is analytic throughout the z-plane except for the points $z = 2$ and $z = 3$.

(a) The function $f(z)$ is analytic in the circle $|z| < 2$ and, therefore, on the basis of Theorem 1 proved in the preceding section, it can be expanded into Taylor's series in powers of z convergent in the circle $|z| < 2$:

$$f(z) = \sum_{k=0}^{\infty} c_k z^k. \tag{8}$$

The numbers c_k can be computed by the formula

$$c_k = \frac{f^{(k)}(0)}{k!} \quad (k = 0, 1, 2, \ldots). \tag{9}$$

However, in the present case series (8) can also be obtained by using the formula for the sum of terms of a decreasing geometric progression. We have (if $|z| < 2$)

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}} = -\frac{1}{3}\left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \ldots\right]$$

$$= -\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k,$$

$$\frac{1}{z-2}=-\frac{1}{2}\,\frac{1}{1-\frac{z}{2}}=-\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\ldots\right]$$

$$=-\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k.$$

Therefore for our function $c_k=\frac{1}{2^{k+1}}-\frac{1}{3^{k+1}}$.

By virtue of uniqueness of the expansion of a function into a power series, the obtained numbers c_k are equal to the corresponding numbers c_k computed by formula (9).

(b) The function $f(z)$ is analytic in the annulus $2<|z|<3$, therefore it can be expanded into the Laurent series

$$f(z)=\sum_{-\infty}^{\infty}c_k z^k, \tag{10}$$

$$c_k=\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)\,d\zeta}{\zeta^{n+1}}\quad(k=0,\ \pm1,\ \pm2,\ \ldots), \tag{11}$$

where γ is the circle $|\zeta|=\rho$, $2<\rho<3$, oriented anticlockwise. But the numbers c_k can be obtained without resorting to complicated formulas (11). For $2<|z|<3$ we have

$$\frac{1}{z-3}=-\sum_{k=0}^{\infty}\frac{z^k}{3^{k+1}},$$

$$\frac{1}{z-2}=\frac{1}{z}\,\frac{1}{1-\frac{2}{z}}=\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{1}{z}\right)^2+\ldots\right]=\sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k}.$$

Therefore the Laurent series of the function $f(z)$ has the form

$$f(z)=-\sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k}-\sum_{k=0}^{\infty}\frac{z^k}{3^{k+1}}.$$

In consequence of the uniqueness of the expansion into the Laurent series, the obtained coefficients are equal to the corresponding numbers c_k determined by formulas (11).

(c) The function $f(z)$ is also analytic in the exterior of the circle $|z| \leqslant 3$, i.e. for the values of z satisfying the inequality $|z| > 3$ and possesses the property

$$\lim_{z \to \infty} f(z) = 0. \tag{12}$$

Therefore $f(z)$ can be expanded in the Laurent series of the form

$$f(z) = \sum_{k=1}^{\infty} \frac{c_{-k}}{z^k}. \tag{13}$$

The terms of the form $c_k z^k$ ($k = 0, 1, \ldots$) cannot enter into the Laurent expansion of the function f, i.e. $c_k = 0$ for the indicated k. Otherwise, this would contradict property (12).

The numbers c_{-k} can also be obtained here directly. For $|z| > 3$ we have

$$\frac{1}{z-3} = \frac{1}{z} \frac{1}{1-\frac{3}{z}} = \frac{1}{z}\left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \ldots\right] = \sum_{k=1}^{\infty} \frac{3^{k-1}}{z^k},$$

$$\frac{1}{z-2} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{z^k}.$$

Therefore

$$f(z) = \sum_{k=1}^{\infty} \frac{3^{k-1} - 2^{k-1}}{z^k}.$$

Example 2. Expand the function

$$f(z) = \frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2} \tag{14}$$

into Taylor's series in powers of $z - i$ and determine the radius of convergence of this series.

Solution. The greatest circle centred at the point i inside which the function $f(z)$ is analytic has the radius equal to the distance of the point i to its nearest singular point. Such point is, obviously, the point $z = 2$. Consequently, the

indicated radius is equal to

$$R = |2 - i| = \sqrt{2^2 + 1^2} = \sqrt{5}.$$

Let us denote by σ an open circle (without the boundary) of radius $R = \sqrt{5}$ centred at the point i.

Inside σ the function $f(z)$ is analytic, and any greater concentric circle contains the singular point $z = 2$ at which the analyticity of the function is violated.

By virtue of Theorem 1 proved in Sec. 6.9, the function $f(z)$ is expanded into Taylor's series in powers of $z - i$. This series is readily obtained in the following manner.

We have

$$\frac{1}{z-2} = \frac{1}{(z-i)+(i-2)} = \frac{1}{\frac{z-i}{i-2}+1} \cdot \frac{1}{i-2}$$

$$= \frac{1}{i-2}\left(1 - \frac{z-i}{i-2} + \left(\frac{z-i}{i-2}\right)^2 - \ldots\right), \quad (15)$$

and we have obtained a power series in powers of $z - i$ obviously convergent in the circle $|z - i| < R$, $R = |i - 2| = \sqrt{5}$. Further,

$$\frac{1}{z-3} = \frac{1}{(z-i)+(i-3)} = \frac{1}{i-3} \frac{1}{1+\frac{z-i}{i-3}}$$

$$= \frac{1}{i-3}\left(1 - \frac{z-i}{i-3} + \left(\frac{z-i}{i-3}\right)^2 - \ldots\right). \quad (16)$$

We have obtained a power series in powers of $z - i$ once again, which also converges in the circle $|z - i| < R$. In fact, it is convergent in the circle of radius $|i - 3| = \sqrt{10}$, but we shall not need this for our further argument.

The difference between series (16) and (15) is the expansion into Taylor's series in powers of $z - i$ of the function $f(z)$. The radius of convergence of this series is equal to $R = \sqrt{5}$.

Problem. Expand the function $f(z)$ (see (14)) into the Laurent series in powers of $z - i$: (a) in an annulus $\sqrt{5} < |z - i| < \sqrt{10}$ and (b) in the neighbourhood of $z = \infty$.

Sec. 6.11. A Classification of Isolated Singular Points. Residues

In the preceding section we proved Theorem 1 which stated that if $0 \leqslant r < R \leqslant \infty$ and a function $f(z)$ is analytic in the annulus

$$r < |z - z_0| < R,$$

then it is expanded into the Laurent series which converges to this function:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k = f_1(z) + f_2(z) \quad (r < |z - z_0| < R), \quad (1)$$

where

$$\left.\begin{aligned} f_1(z) &= \sum_{k=0}^{\infty} c_k (z - z_0)^k \quad (|z - z_0| < R), \\ f_2(z) &= \sum_{k=1}^{\infty} \frac{c_{-k}}{(z - z_0)^k} \quad (|z - z_0| > r). \end{aligned}\right\} \quad (2)$$

Let $r = 0$. It is thus assumed that the function is analytic in the open circle $0 < |z - z_0| < R$, with the point z_0 punctured. At the point z_0 itself the function f is most frequently not defined.

In this case z_0 is said to be an *isolated singular point* of the function f. Below we shall give a classification of isolated singular points.

The power series

$$f_1(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \quad (|z - z_0| < R)$$

has the radius of convergence $R > 0$. Therefore its sum has a continuous derivative in the circle $|z - z_0| < R$.

Consider three cases (for $r = 0$!).

Case (a). The function $f(z)$ has the form

$$f(z) = f_1(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k, \quad (3)$$

i.e. all the numbers $c_{-k} = 0$ ($k = 1, 2, \ldots$). Since power series (3) is convergent for all z with $|z - z_0| < R$, its

radius of convergence is equal to R and, consequently (see Sec. 6.9), its sum $f_1(z)$ is defined and continuously differentiable at all the points of the circle $|z - z_0| < R$ including the point z_0. Hence, the function $f_1(z)$ is analytic in this circle.

Therefore, if we set

$$f(z_0) = f_1(z_0) = c_0,$$

then the function $f(z)$ will also be analytic in this circle.

In this case we say that the *singularity of the function f at the point z_0 is removable.* It suffices to set $f(z_0) = c_0$ for the function f to become analytic not only in the vicinity of the point z_0, but at the point itself.

Note that in this case the integral

$$\int_L f(z)\,dz = 0 \tag{4}$$

for any closed contour L containing an interior point z_0 and belonging to the circle $|z - z_0| < R$.

Case (b). The function $f(z)$ has the form

$$f(z) = f_1(z) + \sum_{k=-1}^{\infty} \frac{c_{-k}}{(z-z_0)^k} = \sum_{k=-m}^{\infty} c_k (z-z_0)^k \quad (c_{-m} \neq 0). \tag{5}$$

Thus, $c_k = 0$ for $k = -(m+1), -(m+2), \ldots$.

In this case the point z_0 is called a *pole of order m of the function f (z).* For $m = 1$ the point z_0 is also termed a *simple pole.*

Since

$$\lim_{z \to z_0} f_1(z) = c_0$$

and

$$\lim_{z \to z_0} \sum_{k=1}^{m} \frac{c_{-k}}{(z-z_0)^k} = \lim_{z \to z_0} \frac{1}{(z-z_0)^m} [c_{-m} + c_{-(m-1)}(z-z_0) + \ldots + c_{-1}(z-z_0)^{m-1}] = \infty, \tag{6}$$

then

$$\lim_{z \to z_0} f(z) = \infty. \tag{7}$$

Now, if L is an anticlockwise oriented contour containing z_0 and belonging to the circle $|z - z_0| < R$, then

$$\int_L f(z)\,dz = 2\pi i c_{-1}. \tag{8}$$

Indeed,

$$\int_L f(z)\,dz = \int_L f_1(z)\,dz + \sum_{k=1}^{m} \int_L \frac{c_{-k}}{(z-z_0)^k}\,dz = 0 + 2\pi i c_{-1},$$

since

$$\int_L \frac{dz}{z-z_0} = 2\pi i, \quad \int_L \frac{dz}{(z-z_0)^k} = 0 \quad (k = 2, \ldots, m)$$

(see (10), (11), Sec. 6.6).

Case (c). The function $f(z)$ has the form

$$f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k + \sum_{k=1}^{\infty} \frac{c_{-k}}{(z-z_0)^k} = f_1(z) + f_2(z), \tag{9}$$

where in the series

$$f_2(z) = \sum_{k=1}^{\infty} \frac{c_{-k}}{(z-z_0)^k}$$

an infinite number of coefficients c_{-k} is not equal to zero.

In this case the function $f(z)$ is said to have an *essential singularity* at the point z_0.

We know that

$$\lim_{z \to z_0} f_1(z) = c_0.$$

But under the indicated conditions $f_2(z)$ does not tend to a limit, finite or infinite, as $z \to z_0$. This fact follows from Sokhotsky's theorem*.

Notice that the arguments mentioned when proving equality (6) in the case of a pole are not applicable here, since the operation of termwise passage to the limit is not always legitimate for infinite sums.

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* Sokhotsky, Yu. V. (1842-1929). Russian mathematician.

Example 1. The function $e^{-1/z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{-2n}$ has an essential singularity at the point $z = 0$. This function has no limit at this point.

Indeed, for $z = x$ (x real) $\exp(-1/x^2) \to 0$, as $x \to 0$. But if $z = \frac{i}{n}$, then $\exp(-1/z^2) = \exp(n^2) \to +\infty$ as $n \to \infty$. Hence, the function $\exp(-1/z^2)$ has no limit at the point $z = 0$.

For any anticlockwise oriented contour L belonging to the circle $|z - z_0| < R$ and containing an interior point z_0, the same as in the case of the pole

$$\int_L f(z)\,dz = 2\pi i c_{-1}. \tag{10}$$

The thing is that in the present case (see Sec. 6.6, remark 2) the integral along L may be replaced by an integral taken round an anticlockwise oriented circle γ centred at the point z_0 belonging to the circle $|z - z_0| < R$. But on γ series (9) are uniformly convergent and, consequently, they can be integrated termwise along γ. But, as we know,

$$\int_\gamma \frac{dz}{(z-z_0)^k} = 0 \quad (k \neq 1) \quad \text{and} \quad \int_\gamma \frac{dz}{z-z_0} = 2\pi i,$$

which yields equality (10).

Let us now give the following definition: *Let z_0 be an isolated singular point of the function $f(z)$, i. e. let the function $f(z)$ be analytic in a circle*

$$|z - z_0| < R,$$

with the point z_0 punctured. The residue of the function f at the point z_0 is the integral

$$\frac{1}{2\pi i} \int_L f(z)\,dz = \operatorname*{Res}_{z=z_0} f(z), \tag{11}$$

where L is a contour contained in the circle $|z - z_0| < R$, anticlockwise oriented and containing the point z_0.

On the basis of the aforesaid (see Cases (a), (b), (c)), if

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k \quad (0 < |z - z_0| < R)$$

is the Laurent series of f at the point z_0, then

$$\operatorname*{Res}_{z=z_0} f(z) = c_{-1}. \tag{12}$$

Therefore, with the expansion of a function in the Laurent series known, the residue is readily found at any singular point.

In particular, if z_0 is a removable singularity, then $\operatorname*{Res}_{z=z_0} f(z) = 0$.

It is sometimes difficult to expand the function $f(z)$ into the Laurent series and therefore it is necessary to look for some other methods not involving the Laurent expansion of the given function.

Let $z = z_0$ be a pole of order $m \geqslant 1$. Then

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k + \sum_{k=1}^{m} c_{-k} (z - z_0)^{-k} \quad (c_{-m} \neq 0). \tag{13}$$

Multiplying both sides of (13) by $(z - z_0)^m$, we have

$$(z - z_0)^m f(z) = c_{-m} + c_{-m+1}(z - z_0) + \ldots + c_{-1}(z - z_0)^{m-1} + \sum_{k=0}^{\infty} c_k (z - z_0)^{k+m}. \tag{14}$$

If equality (14) is differentiated $(m - 1)$ times, then the constant term on the right will be equal to $(m - 1)!c_{-1}$ and consequently,

$$\lim_{z \to z_0} \frac{d^{m-1}[(z - z_0)^m f(z)]}{dz^{m-1}} = (m - 1)!\, c_{-1},$$

whence

$$\operatorname*{Res}_{z=z_0} f(z) = c_{-1} = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}[(z - z_0)^m f(z)]}{dz^{m-1}}. \tag{15}$$

If the function $f(z) = \dfrac{\varphi(z)}{\psi(z)}$, where $\varphi(z_0) \neq 0$, and $\psi(z)$ has a simple zero for $z = z_0$ ($\psi(z_0) = 0$, $\psi'(z_0) \neq 0$), then $z = z_0$ is a simple pole of $f(z)$. On the basis of formula (15) (for $m = 1$), we have

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{\varphi(z)}{\psi(z)} = \lim_{z \to z_0} (z - z_0) \frac{\varphi(z)}{\psi(z)} = \lim_{z \to z_0} \frac{\varphi(z)}{\dfrac{\psi(z) - \psi(z_0)}{z - z_0}} = \frac{\varphi(z_0)}{\psi'(z_0)}.$$

Thus, in the present case

$$\operatorname*{Res}_{z=z_0} f(z) = c_{-1} = \frac{\varphi(z_0)}{\psi'(z_0)}. \tag{16}$$

In the case when $z = z_0$ is an essential singular point there is only one method for evaluating the residue — expanding the function $f(z)$ in the Laurent series.

Example 2. Find the residue of the function $f(z) = \frac{\sin^2 z}{\cos z}$ at the point $z = \frac{\pi}{2}$.

In the present case $f(z) = \frac{\varphi(z)}{\psi(z)}$, where $\varphi(z) = \sin^2 z$, $\psi(z) = \cos z$. The point $z = \pi/2$ is a simple pole of the function $f(z)$, since $\varphi(\pi/2) = 1 \neq 0$, $\psi(\pi/2) = 0$, $\psi'(z) = -\sin z$, $\psi'(\pi/2) = -1 \neq 0$. Hence, by formula (16), we obtain

$$\operatorname*{Res}_{z=\pi/2} f(z) = \frac{\varphi(\pi/2)}{\psi'(\pi/2)} = \frac{1}{-1} = -1.$$

Example 3. Find the residue of the function $\exp(1/z)$ at the point $z = 0$.

We have

$$\exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \ldots .$$

Hence, $z = 0$ is an essential singular point and

$$\operatorname*{Res}_{z=0} \exp\left(\frac{1}{z}\right) = c_{-1} = 1.$$

Example 4. Find the residue of the function

$$f(z) = \frac{1}{(z-2)^2(z-3)}$$

about the point $z = 2$.

The given point is a pole of the second order, therefore, by formula (15), we have

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} \frac{d}{dz}\left[(z-2)^2 f(z)\right] = \lim_{z\to 2} \frac{d}{dz}\,\frac{1}{z-3}$$

$$= \lim_{z\to 2} \frac{-1}{(z-3)^2} = -1.$$

Sec. 6.12. A Classification of Singular Points at Infinity

Let us now suppose that in Theorem 1 of Sec. 6.10 $z_0 = 0$ and $R = \infty$, and r is any nonnegative number ($0 \leqslant r < \infty$). Then Theorem 1 states: if the function $f(z)$ is analytic for all complex numbers z satisfying the inequality

$$|z| > r, \tag{1}$$

then it can be expanded into the Laurent series in powers of z:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k = F_1(z) + F_2(z) \quad (|z| > r), \tag{2}$$

convergent for all z with $|z| > r$. Here

$$F_1(z) = \sum_{k=0}^{\infty} \frac{c_{-k}}{z^k}, \quad F_2(z) = \sum_{k=1}^{\infty} c_k z^k. \tag{3}$$

Set (1) is called the exterior of the circle $|z| \leqslant r$. It is convenient to regard this set as the neighbourhood of a point at infinity (point ∞).

Thus, we add formally an abstract point at infinity ($z = \infty$) to the set of complex points (numbers).

The function $f(z)$ is analytic in the neighbourhood of the point $z = \infty$ except for the point ∞ itself which is naturally called in this case an isolated singular point of the function f.

Depending on the behaviour of the function $f(z)$ in the neighbourhood of the point $z = \infty$, it is natural to introduce the following classification:

(a) The singularity at the point $z = \infty$ is removable if

$$c_k = 0 \quad (k = 1, 2, \ldots),$$

i.e. if

$$f(z) = F_1(z) \quad (|z| > r).$$

In this case

$$\lim_{z \to \infty} f(z) = \lim_{z \to \infty} F_1(z) = c_0.$$

It is also obvious that

$$\frac{1}{2\pi i}\int_{L_-} f(z)\,dz = -c_{-1},$$

where LA is an arbitrary clockwise oriented contour containing entirely the circle $|z| = r$ (Fig. 143). Here it is suitable

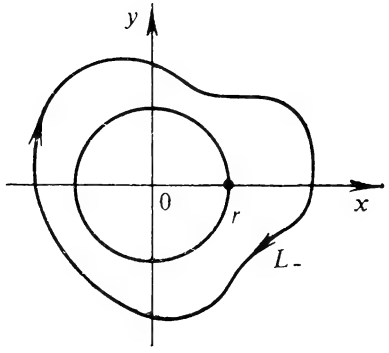

Fig. 143.

to imagine that the point ∞ is found inside the contour L_- — the contour L_- "bypasses" the point ∞ leaving the latter on its left.

(b) The point $z = \infty$ is a pole of order m if

$$f(z) = F_1(z) + \sum_{k=1}^{m} c_k z^k$$

$$(c_m \neq 0).$$

In this case, obviously, $\lim\limits_{z\to\infty} f(z) = \infty$. Further,

$$\int_{L_-} f(z)\,dz = \sum_{k=0}^{\infty} c_{-k}\int_{L_-}\frac{dz}{z^k} + \sum_{k=1}^{m} c_k\int_{L_-} z^k\,dz$$

$$= -c_{-1}\int_{L}\frac{dz}{z} = -2\pi i c_{-1},$$

since

$$\int_{L_-} z^k\,dz = -\int_{L} z^k\,dz = 0 \quad (k \neq -1).$$

(c) $z = \infty$ is an essential singular point if

$$f(z) = F_1(z) + \sum_{k=1}^{\infty} c_k z^k \tag{4}$$

and there is an infinitude of nonzero numbers c_k.

The function $F_1(z)$ tends to a finite limit as $z \to \infty$, while the function $\sum_{k=1}^{\infty} c_k z^k$, by Sokhotsky's theorem, does not tend to any limit as $z \to \infty$. Therefore the function $f(z)$ does not tend to a limit either.

Further,

$$\int_{L_-} f(z)\,dz = \sum_{k=-\infty}^{\infty} c_k \int_{L_-} z^k\,dz = -2\pi i c_{-1}.$$

Termwise integration is legitimate here, since, as we know, integrals $\int_{L_-}$ can be replaced by integrals $\int_{G}$ taken round the circle of radius $\rho > r$ on which series (4) is uniformly convergent.

Let us introduce the following definition.

The residue of an analytic function $f(z)$ at a point at infinity is defined as

$$\frac{1}{2\pi i} \int_{L_-} f(z)\,dz = \operatorname*{Res}_{z=\infty} f(z),$$

where L_- is an arbitrary closed contour, oriented clockwise and belonging to the set $|z| > r$. In this case we say that when the contour is traversed clockwise, "the point ∞ remains on the left".

On the basis of the aforesaid (see (a), (b), (c)), if

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k \quad (|z| > r),$$

is the Laurent series of the function f in the exterior of the circle $|z| = r$, then

$$\operatorname*{Res}_{z=\infty} f(z) = -c_{-1}.$$

If $z = \infty$ is a removable singularity, then there are no positive powers of z in the Laurent series of the function $f(z)$, whereas z^{-1} can enter into it; therefore, in this case $\operatorname{Res}\limits_{z=\infty} f(z)$ is not necessarily equal to zero.

Example 1. The function

$$f(z) = \frac{z}{z^2+1} = \frac{1}{z\left(1+\frac{1}{z^2}\right)}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{-2n} = \sum_{n=0}^{\infty}(-1)^n z^{-1-2n} \quad (|z|>1)$$

has a removable singularity at the point $z = \infty$ and $c_{-1} = 1$, hence

$$\operatorname{Res}_{z=\infty} f(z) = -1.$$

Sec. 6.13. The Residue Theorem

Theorem 1. *Let the function $f(z)$ be analytic in the extended complex plane z, with the exception of a finite number of points $z_1, \ldots, z_N$. Then*

$$\sum_{k=1}^{N} \operatorname{Res}_{z=z_k} f(z) + \operatorname{Res}_{z=\infty} f(z) = 0. \quad (1)$$

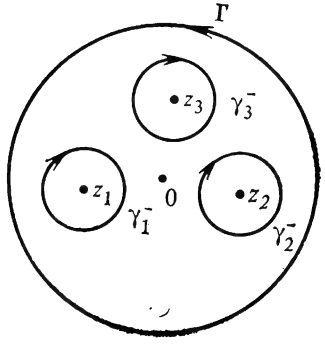

Fig. 144.

Proof. Let us construct circles $\gamma_1^-, \gamma_2^-, \ldots, \gamma_N^-$, oriented clockwise and centred at $z_1, z_2, \ldots, z_N$, respectively, of radii so small as to prevent their intersection. We then construct a circle Γ oriented anticlockwise, centred at the zero point and of radius as great as to enclose the circles $\gamma_1^-, \ldots, \gamma_N^-$ (Fig. 144). The composite contour $L = \gamma_1^- + \gamma_2^- + \ldots + \gamma_N^- + \Gamma$ bounds the domain Ω inside which the function $f(z)$ is analytic. It is also analytic on L. When traversing along L the domain Ω remains on the left.

But then, by Cauchy's theorem for a composite contour,

$$\int\limits_{\gamma_1^-} f(z)\,dz + \ldots + \int\limits_{\gamma_N^-} f(z)\,dz + \int\limits_{\Gamma} f(z)\,dz = 0 \qquad (2)$$

or, multiplying the left-hand side of this equality by $1/2\,\pi i$, we obtain

$$\frac{1}{2\pi i}\int\limits_{\gamma_1} f(z)\,dz + \ldots + \frac{1}{2\pi i}\int\limits_{\gamma_N} f(z)\,dz + \frac{1}{2\pi i}\int\limits_{\Gamma_-} f(z)\,dz = 0,$$

i.e.

$$\operatorname*{Res}_{z=z_1} f(z) + \ldots + \operatorname*{Res}_{z=z_N} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

It should be taken into account that inside each of the contours γ_k there is only one singular point z_k, and outside Γ only one singular point $z = \infty$. The theorem has been proved.

The application of this theorem is reduced to the following. If it is difficult to compute one of the integrals entering into (2), then we can try to compute the rest of the integrals entering into (2) and to obtain the required integral from (2).

The computation of these integrals itself is reduced to expanding the function $f(z)$ into the Laurent series in the neighbourhoods of the corresponding singular points. Generally speaking, one need not know these expansions completely; it is sufficient to know only the terms of the form $c_{-1}/(z - z_k)$ entering into these expansions to obtain the required results.

Sec. 6.14. Evaluation of Integrals with the Aid of Residues

Let the function $f(z)$ be analytic everywhere in the upper half-plane, including the real axis, with the exception of a finite number of singular points $a_1, a_2, \ldots, a_N$, lying in the upper half-plane. Under these conditions we shall consider the methods of evaluation of the integrals

$$\int\limits_{-\infty}^{\infty} f(x)\,dx, \qquad \int\limits_{-\infty}^{\infty} f(x)\,e^{ix}\,dx.$$

Theorem 1. *Let the function $f(z)$ satisfy the above mentioned conditions and, in addition, $|f(z)| \leqslant M/|z|^m$ for $|z| \geqslant R$, where $m \geqslant 2$ and R is a sufficiently large number. Then*

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{j=1}^{N} \operatorname*{Res}_{z=a_j} f(z). \tag{1}$$

Proof. Let us describe a semicircle L (oriented anticlockwise) of radius R with centre at the point O so that all the

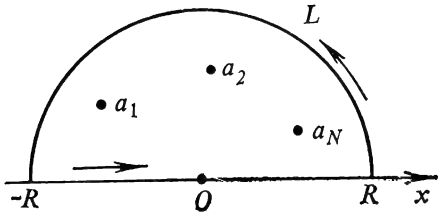

Fig. 145.

singular points of the function $f(z)$ are enclosed inside L (Fig. 145). By virtue of Theorem 1 proved in the preceding section,

$$\int_{-R}^{R} f(x)\,dx + \int_{L} f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname*{Res}_{z=a_j} f(z). \tag{2}$$

Since $|f(z)| \leqslant M/|z|^m$ for $|z| \geqslant R$, we have

$$\left|\int_{L} f(z)\,dz\right| \leqslant \frac{M}{R^m}\,\pi R = \frac{\pi M}{R^{m-1}} \to 0, \quad R \to \infty \quad (m-1 \geqslant 1).$$

Passing to the limit in equality (2) as $R \to \infty$, we obtain (1)

Example 1. Compute the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

The function $f(z) = \frac{1}{1+z^4}$ is analytic in the upper half-plane except for the points

$$a_1 = e^{\pi i/4} = \frac{\sqrt{2}}{2}(1+i), \quad a_2 = e^{3\pi i/4} = \frac{\sqrt{2}}{2}(i-1),$$

at which it has simple poles. Besides, $|f(z)| \leqslant 1/|z|^4$ ($m = 4 > 2$). Let us find the residues of the function $f(z)$ at the points a_1, a_2. By formula (16) derived in Sec. 6.11,

$$\operatorname*{Res}_{z=a_j} f(z) = \frac{1}{\psi'(a_j)} \quad (j = 1, 2),$$

where $\psi(z) = 1 + z^4$. We have $\psi'(z) = 4z^3$, $\psi'(a_1) = 4e^{3\pi i/4} = -4e^{-i\pi/4} \neq 0$, $\psi'(a_2) = 4e^{9\pi i/4} = 4e^{i\pi/4} \neq 0$.

Hence,

$$\operatorname*{Res}_{z=a_1} f(z) = -\frac{1}{4} e^{i\pi/4}, \quad \operatorname*{Res}_{z=a_2} f(z) = \frac{1}{4} e^{-i\pi/4}.$$

By formula (1), we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{2\pi i}{4}\left(-e^{i\pi/4} + e^{-i\pi/4}\right) = \pi \frac{e^{i\pi/4} - e^{-i\pi/4}}{2i}$$

$$= \pi \sin\frac{\pi}{4} = \pi \frac{\sqrt{2}}{2}.$$

Theorem 2. *Let the function $f(z)$ satisfy the conditions mentioned at the beginning of the present section and* $\lim\limits_{z\to\infty} f(z) = 0$ *uniformly with respect to* $\arg z = \varphi$. *Then*

$$\int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx = 2\pi i \sum_{j=1}^{N} \operatorname*{Res}_{z=a_j} f(z)\, e^{iz}. \tag{3}$$

Proof. The same as in the proof of Theorem 1, we have

$$\int_{-R}^{R} f(x)\, e^{ix}\, dx + \int_{L} f(z)\, e^{iz}\, dz = 2\pi i \sum_{j=1}^{N} \operatorname*{Res}_{z=a_j} f(z)\, e^{iz} \tag{4}$$

(the function $f(z)e^{iz}$ has the same singularities as $f(z)$).

We have to prove that the integral $\int\limits_L f(z)e^{iz}\, dz$ tends to zero as $R \to \infty$. We have

$$\Lambda = \left|\int_L f(z)\, e^{iz}\, dz\right| = \left|\int_0^{\pi} f(Re^{i\varphi})\, e^{-R\sin\varphi} e^{iR\cos\varphi} iRe^{i\varphi}\, d\varphi\right|$$

$$\leqslant \int_0^{\pi} |f(Re^{i\varphi})|\, e^{-R\sin\varphi} R\, d\varphi.$$

By virtue of the conditions of the theorem, $|f(Re^{i\varphi})| \leqslant \varepsilon$ for $R > N_1$ for all φ $(0 \leqslant \varphi \leqslant \pi)$. Therefore ($\sin\varphi > 2\varphi/\pi$ for $0 \leqslant \varphi \leqslant \pi/2$)

$$\Lambda \leqslant \varepsilon \int_0^{\pi} Re^{-R\sin\varphi}\, d\varphi$$

$$= 2\varepsilon \int_0^{\pi/2} Re^{-R\sin\varphi}\, d\varphi \leqslant 2\varepsilon \int_0^{\pi/2} Re^{-2R\varphi/\pi}\, d\varphi$$

$$= \left(\frac{2R}{\pi}\varphi = t\right) = \varepsilon\pi \int_0^{R} e^{-t}\, dt = \pi\varepsilon\,(1 - e^{-R}) < \pi\varepsilon \quad (R > N_1).$$

Passing to the limit in (4) as $R \to \infty$, we obtain (3).

If the function $f(z)$ has singularities on the real axis, then, by means of a special construction of the integration contour,

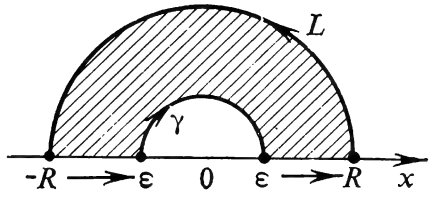

Fig. 146.

we can evaluate the corresponding integrals, provided they exist.

Example 2. Let $f(z) = 1/z$. This function has a simple pole on the real axis at the point $z = 0$. Further, $\lim\limits_{z\to\infty} f(z) = 0$ uniformly relative to $\arg z = \varphi$.

Let us construct the contour of integration as in Fig. 146. The contour is traversed in the direction indicated by the arrows. In the hatched portion the function e^{iz}/z is analytic for any R and ε, therefore, by Cauchy's theorem (the semicircle γ is oriented anticlockwise),

$$\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \frac{e^{ix}}{x}\, dx + \int_L \frac{e^{iz}}{z}\, dz - \int_\gamma \frac{e^{iz}}{z}\, dz = 0. \qquad (5)$$

As above, it is easy to show that $\lim\limits_{R\to\infty}\int\limits_{L} z^{-1}e^{iz}\,dz=0$.

Further,

$$\lim_{\varepsilon\to 0}\int\limits_{\gamma} e^{iz}\frac{dz}{z}=\lim_{\varepsilon\to 0}\int\limits_{0}^{\pi} e^{i\varepsilon(\cos\varphi+i\sin\varphi)}\frac{i\varepsilon e^{i\varphi}\,d\varphi}{\varepsilon e^{i\varphi}}$$

$$=i\lim_{\varepsilon\to 0}\int\limits_{0}^{\pi} e^{i\varepsilon(\cos\varphi+i\sin\varphi)}\,d\varphi=i\int\limits_{0}^{\pi} d\varphi=\pi i.$$

Thus, equality (5) in the limit, as $R\to\infty$ and $\varepsilon\to 0$, takes the form

$$\pi i=\lim_{\substack{\varepsilon\to 0\\ R\to\infty}}\left(\int\limits_{-R}^{-\varepsilon}+\int\limits_{\varepsilon}^{R}\right)\frac{e^{ix}}{x}\,dx$$

$$=\lim_{\substack{\varepsilon\to 0\\ R\to\infty}}2i\int\limits_{\varepsilon}^{R}\frac{e^{ix}-e^{-ix}}{2xi}\,dx=2i\int\limits_{0}^{\infty}\frac{\sin x}{x}\,dx,$$

i.e.

$$\int\limits_{0}^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}.$$

Since the function $\frac{\sin x}{x}$ is even,

$$\int\limits_{-\infty}^{\infty}\frac{\sin x}{x}\,dx=\pi.$$

Remark 1. If there is the factor $\sin x$ or $\cos x$ under the integral sign, then it is frequently convenient to consider the integral of a function in which $\sin x$ or $\cos x$ is replaced by e^{iz}. It is explained by the fact that $|\sin z|$ and $|\cos z|$ increase unboundedly as $z\to\infty$, and $|e^{iz}|=e^{-y}\to 0$ for $y\to\infty$ $(y>0)$. Therefore the function $f(z)\begin{Bmatrix}\sin z\\ \cos z\end{Bmatrix}$ will behave, generally speaking, differently as compared with the function $f(z)$. Then, on evaluating the integral

$\int\limits_{-\infty}^{\infty} f(x)e^{ix}dx$, we find $\int\limits_{-\infty}^{\infty} f(x)\cos x\,dx$ and $\int\limits_{-\infty}^{\infty} f(x)\sin x\,dx$ (by separating the real and imaginary parts).

Example 3. Evaluate the integral

$$\int\limits_{0}^{\infty} \frac{\cos \alpha x\,dx}{a^2+x^2} \quad (\alpha,\ a>0).$$

Consider the function $e^{i\alpha z}/(a^2+z^2)$. This function is analytic in the upper half-plane except the point $z = ai$. The function $f(z) = 1/(a^2+z^2) \to 0$ as $z \to \infty$ uniformly with respect to $\arg z = \varphi$. Therefore, by Theorem 2,

$$\int\limits_{-\infty}^{\infty} \frac{e^{i\alpha x}}{a^2+x^2}\,dx = 2\pi i \operatorname*{Res}_{z=ai} \frac{e^{i\alpha z}}{a^2+z^2} = 2\pi i\,\frac{e^{i\alpha ai}}{2ai} = \frac{\pi}{a}\,e^{-\alpha a}.$$

Separating the real part, we obtain

$$\int\limits_{-\infty}^{\infty} \frac{\cos \alpha x}{a^2+x^2}\,dx = \frac{\pi}{a}\,e^{-\alpha a}, \quad \int\limits_{0}^{\infty} \frac{\cos \alpha x}{a^2+x^2}\,dx = \frac{\pi}{2a}\,e^{-\alpha a}.$$

Example 4. Compute the integral

$$I = \int\limits_{0}^{\infty} \frac{\sin^2 x}{1+x^2}\,dx.$$

We have

$$I = \int\limits_{0}^{\infty} \frac{1-\cos 2x}{2(1+x^2)}\,dx = \frac{1}{2}\int\limits_{0}^{\infty} \frac{dx}{1+x^2} - \frac{1}{2}\int\limits_{0}^{\infty} \frac{\cos 2x}{1+x^2}\,dx$$

$$= \frac{1}{2}\arctan x\Big|_0^{\infty} - \frac{1}{2}\,\frac{\pi}{2}\,e^{-2} = \frac{\pi}{4} - \frac{\pi}{4}\,e^{-2} = \frac{\pi}{4}(1-e^{-2}).$$

Thus,

$$\int\limits_{0}^{\infty} \frac{\sin^2 x\,dx}{1+x^2} = \frac{\pi}{4}(1-e^{-2}).$$

Example 5.

$$\int_0^\infty \frac{\cos^2 x\,dx}{1+x^2} = \int_0^\infty \frac{1-\sin^2 x}{1+x^2}\,dx$$

$$= \int_0^\infty \frac{dx}{1+x^2} - \int_0^\infty \frac{\sin^2 x}{1+x^2}\,dx = \frac{\pi}{2} - \frac{\pi}{4}(1-e^{-2}) = \frac{\pi}{4}(1+e^{-2}).$$

Example 6. Compute the Fresnel* integrals

$$\int_{-\infty}^{\infty} \cos x^2\,dx, \qquad \int_{-\infty}^{\infty} \sin x^2\,dx.$$

Consider the function e^{iz^2}. This function is analytic in

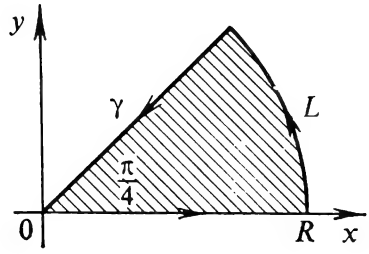

Fig. 147.

the hatched domain (Fig. 147), therefore, by Cauchy's theorem,

$$\int_0^R e^{ix^2}\,dx + \int_L e^{iz^2}\,dz + \int_\gamma e^{iz^2}\,dz = 0,$$

where L is a part of the circle $|z| = R$, and γ is a segment of the straight line $z = \rho e^{i\pi/4}$, $0 \leqslant \rho \leqslant R$ (oriented accord-

* Fresnel, Augustin Jean (1788-1827). French physicist and engineer.

ing to the arrows). Further,

$$\int_{\gamma} e^{iz^2}\,dz = \int_{R}^{0} e^{i\rho^2 i} e^{i\pi/4}\,d\rho = -e^{i\pi/4}\int_{0}^{R} e^{-\rho^2}\,d\rho \to$$

$$\to -e^{i\pi/4}\int_{0}^{\infty} e^{-\rho^2}\,d\rho;$$

$$\left|\int_{L} e^{iz^2}\,dz\right| = \left|\int_{0}^{\pi/4} e^{i(Re^{i\varphi})^2} Rie^{i\varphi}\,d\varphi\right|$$

$$= \left|\int_{0}^{\pi/4} Rie^{i\varphi} e^{iR^2\cos 2\varphi} e^{-R^2\sin 2\varphi}\,d\varphi\right| \leqslant \int_{0}^{\pi/4} Re^{-R^2\sin 2\varphi}\,d\varphi$$

$$\leqslant R\int_{0}^{\pi/4} e^{-R^2\frac{4\varphi}{\pi}}\,d\varphi \leqslant \frac{c}{R}\int_{0}^{\infty} e^{-t}\,dt \to 0, \quad R \to \infty.$$

Thus, in the limit as $R \to \infty$ we get (see Sec. 2.13, Example 3)

$$\int_{0}^{\infty} e^{ix^2}\,dx = e^{i\pi/4}\int_{0}^{\infty} e^{-\rho^2}\,d\rho = e^{\pi i/4}\sqrt{\frac{\pi}{4}}.$$

Separating the real and imaginary parts, we obtain

$$\int_{0}^{\infty} \cos x^2\,dx = \frac{\sqrt{2}}{2}\sqrt{\frac{\pi}{4}}, \qquad \int_{0}^{\infty} \sin x^2\,dx = \frac{\sqrt{2}}{2}\sqrt{\frac{\pi}{4}},$$

i.e.

$$\int_{0}^{\infty} \cos x^2\,dx = \int_{0}^{\infty} \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} = \sqrt{\frac{\pi}{8}}.$$

Sec. 6.15. Linear Functions. A Linear Fractional Function

Linear Entire (Integral) Function. Let us consider three functions

$$w = z + c, \tag{1}$$

$$w = rz, \tag{2}$$

$$w = e^{i\theta}z, \tag{3}$$

where c is a constant complex number, $r > 0$, and θ is an arbitrary real number.

Each of these functions maps the z-plane onto the extended w-plane.

Function (1) brings about a simple shear of the z-plane by the vector c (Fig. 148).

Function (2) ($r > 0$!) effects r-fold elongations (for $r > 1$) and compressions (for $r < 1$) of the z-plane: $|w| = r|z|$,

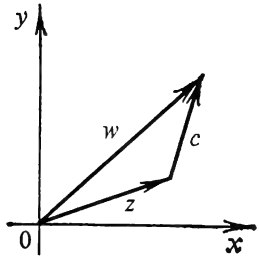

Fig. 148.

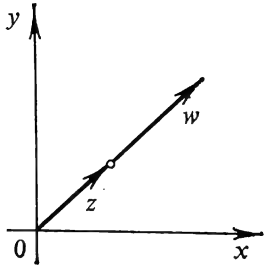

Fig. 149.

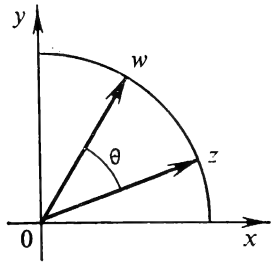

Fig. 150.

Arg w = Arg z. Figure 149 illustrates the case when $r > 1$.

Function (3) realizes a rotation of the z-plane about the origin through an angle θ (Fig. 150).

Functions (1), (2), (3) have respective nonzero derivatives

$$w' = 1, \quad w' = r, \quad w' = e^{i\theta},$$

and therefore they realize conformal transformations.

The three functions under consideration are particular cases of a more general linear entire function

$$w = az + b \quad (a \neq 0), \tag{4}$$

where a and b are constant complex numbers.

The transformation represented by this function can be written in the form $w = a(z + c) = re^{i\theta}(z+c)$, where $c = b/a$, $a = re^{i\theta}$.

Hence it follows that it is reduced to (1), (2), (3):

$$w = ru, \quad u = e^{i\theta}v, \quad v = z + c.$$

In other words, the transformation of the z-plane effected by function (4) is reduced to a shear (by the vector c), then

to a rotation of the plane (through an angle θ), and then to an r-fold stretching or compression of the plane.

The *function* $w = \frac{1}{z}$. Putting $z = re^{i\varphi}$, $w = \rho e^{i\theta}$, we have

$$\rho = \frac{1}{r}, \quad \theta = -\varphi, \tag{5}$$

where the second equality should be understood with an accuracy to $2k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$).

Hence it is seen that the circle $|z| = 1$ goes into itself or, which is more precise, each of its points goes into a point symmetric about the real axis.

Note that if the circle $|z| = 1$ is traversed anticlockwise, then the mapped circle $|w| = 1$ is traversed clockwise.

Transformation (5) is suitable to be split into two transformations

$$r' = \frac{1}{r}, \quad \varphi' = \varphi; \tag{6}$$

$$\rho = r', \quad \theta = -\varphi'. \tag{7}$$

Transformation (6) is called inversion with respect to the unit circle.

The inversion with respect to the unit circle brings the points z and z' lying on the ray making an angle φ with the x-axis into the points lying on the same ray, so that

$$rr' = 1.$$

How to construct the point $z' = r'e^{i\varphi}$ given the point $z = re^{i\varphi}$ is shown in Fig. 151 where the point z lies outside the circle $|z| = 1$. From the point z draw a tangent line to the circle $|z| = 1$, T being the point of tangency; Tz' is perpendicular to Oz. Since the triangle OTz' is similar to the triangle OTz, we have

$$\frac{OT}{r'} = \frac{r}{OT}, \quad OT = 1, \quad rr' = 1.$$

If the point z is found inside the circle $|z| = 1$, then we erect at it a perpendicular to Oz to intersect the circle at the point T. Through the latter draw a tangent line to the circle to intersect the ray Oz. The point of intersection is just the point z'.

The points z and z' are said to be mutually symmetric about the circle $|z| = 1$.

Mapping now (according to (7)) the point z' in a mirror-like manner about the real axis, we obtain the point

$$w = r'e^{-i\varphi} = \frac{1}{re^{i\varphi}} = \frac{1}{z}.$$

It is seen from the formula $w = 1/z$ that, as $z \to 0$, the point w has an unboundedly increasing modulus, therefore it

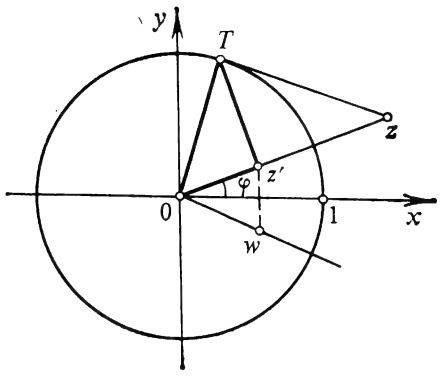

Fig. 151.

is convenient to assume that, with the aid of this formula, to the point $z = 0$ there corresponds a point at infinity which is denoted by the symbol $w = \infty$.

Thus, the function $w = 1/z$ maps the z-plane onto the w-plane by means of inversion about the circle $|z| = 1$ and mirror-like mapping about the x-axis. Here the point $z = 0$ goes into the point $w = \infty$, and the point $z = \infty$ into the point $w = 0$.

Further, $w' = \frac{-1}{z^2} \neq 0$ for any z with $|z| > 0$, therefore the mapping brought about by the function $w = \frac{1}{z}$ ($|z| > 0$) is conformal.

Linear Fractional Function

$$w = \frac{az+b}{cz+d}, \tag{8}$$

We shall assume that $ad - bc \neq 0$. Obviously, the point $z = -d/c$ $(c \neq 0)$ goes into the point $w = \infty$.

By singling out its integral part, we can represent the function w in the form

$$w = \frac{a}{c} + \frac{bc - ad}{c(cz + d)}, \tag{9}$$

whence it is seen that

$$w' = \frac{ad - bc}{(cz + d)^2} \neq 0 \quad (cz + d \neq 0),$$

i.e. the mapping by function (8) is conformal.

From equality (9) it is seen that the given mapping consists of the above considered mappings:

$$z' = cz + d, \quad z'' = \frac{1}{z'}, \quad w = Az'' + B.$$

If we regard a straight line as a circle of an infinite radius, then in transformation (9) a circle goes into a circle (so-called circular property).

From geometrical considerations it is clear that in translation, stretching, and rotation a circle goes into a circle. Moreover, the interior of a circle being mapped goes onto the interior of the mapped circle. Therefore it is sufficient to check the circular property for the transformation $w = \frac{1}{z}$.

As we know, the equation of a circle in the xy-plane has the form

$$A(x^2 + y^2) + mx + ny + l = 0$$

or

$$Az \cdot \bar{z} + m\frac{z + \bar{z}}{2} + n\frac{z - \bar{z}}{2i} + l = 0 \quad (z = x + iy,\ \bar{z} = x - iy)$$

or

$$Az\bar{z} + \bar{B}z + B\bar{z} + l = 0 \quad \left(B = \frac{m + in}{2}\right). \tag{10}$$

In the case under consideration $z = 1/w$, $\bar{z} = 1/\bar{w}$. Consequently, equation (10) turns into the equation

$$A\frac{1}{w\bar{w}} + \frac{\bar{B}}{w} + \frac{B}{\bar{w}} + l = 0$$

or into the equation

$$A + \overline{B}\overline{w} + B\dot{w} + lw\overline{w} = 0,$$

which describes a circle in the w-plane.

In particular, for $l = 0$ we obtain a straight line, i.e. a circle passing through the origin in the z-plane is mapped into a straight line in the w-plane.

Notice that the transformation realized by function (9) can map the interior of a circle both onto the interior and exterior of the mapped circle.

In principle, function (9) depends on three parameters which may be taken as the ratio of the numbers a, b, c, d to one of them (not equal to zero).

Therefore, to define transformation (9), three conditions should be set. Usually, three pairs of appropriate points are given:

$$w_k = \frac{a}{c} + \frac{bc-ad}{c(cz_k+d)} \quad (k=1, 2, 3).$$

It is easy to compute that

$$w - w_k = \frac{(ad-bc)(z-z_k)}{(cz+d)(cz_k+d)}, \quad w_k - w_j = \frac{(ad-bc)(z_k-z_j)}{(cz_k+d)(cz_j+d)}.$$

Hence

$$\frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2} = \frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2}. \tag{11}$$

This is just transformation (9) carrying the points z_k into w_k ($k = 1, 2, 3$).

Let there be given two circles Γ and Γ' in the z- and w-planes, respectively. It is required to find the linear fractional mapping carrying Γ onto Γ' and the interior of Γ onto the interior of Γ'.

Let us take arbitrary triples of points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ on Γ and Γ', respectively, following in the positive direction, i.e. anticlockwise. Then transformation (11) will be the solution of the problem in question.

Indeed, it maps the points z_k into points w_k ($k = 1, 2, 3$) and, obviously, the circle Γ onto Γ' (by virtue of the circular property).

The fact that in the given case the interior of Γ goes into the interior of Γ' follows from the conformality of the mapping realized by a linear fractional function.

In the present case the circles Γ, Γ' have a positive orientation (i.e. they are traversed anticlockwise). By virtue of the conformality of the mapping an interior normal to Γ

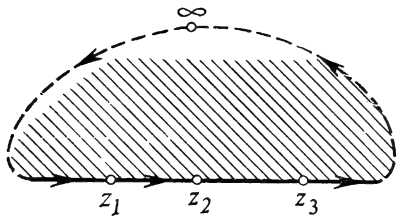

Fig. 152.

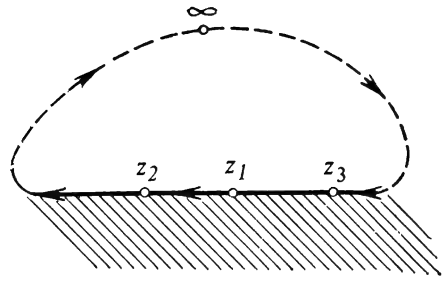

Fig. 153.

(for instance, at the point z_1) goes into a circular arc (perpendicular to Γ' at the point w_1) which is found inside Γ' and this just ensures the mapping of the interior of Γ onto the interior of Γ'.

And if it is required to find the linear fractional transformation which maps Γ onto Γ' and the interior of Γ onto the exterior of Γ', then in formula (11) we must take the points $\{z_1, z_2, z_3\}$ on Γ arranged in the positive direction, and the points $\{w_1, w_2, w_3\}$ on Γ' in the negative direction.

These conclusions are also extended to the case when either Γ or Γ' or both Γ and Γ' are straight lines. But it is required to explain what should be understood by the interior of the straight line Γ when the points $\{z_1, z_2, z_3\}$ are marked on it.

In the case of Fig. 152 this is the upper half-plane, and in the case of Fig. 153 the lower half-plane.

When completed with the point ∞, the straight line Γ may be regarded as a continuous circle of an infinite radius (see Figs. 152 and 153).

We shall move along Γ (possibly, through the point at infinity) from z_1 to z_2 in the direction such that the arc z_1z_2 does not contain z_3. This determines the direction of traverse

on Γ and then the interior of Γ is defined as the domain situated on the left of Γ when moving in this direction. In fact, this domain is the upper or the lower half-plane.

Problems

1. Find the linear fractional transformation which maps the interior of the unit circle onto the upper half-plane so that the points $z_1 = +1$, $z_2 = i$, $z_3 = -1$ go into the points $w_1 = 0$, $w_2 = 1$, $w_3 = 2$, respectively.

2. Write the linear fractional transformation of the upper half-plane into itself which carries the points $z_1 = -1$, $z_2 = 0$, $z_3 = 1$ into the points $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, respectively.

CHAPTER 7

OPERATIONAL CALCULUS

Sec. 7.1. The Laplace Transform

In this chapter, as a rule, we shall consider the functions $f(t)$ of the real variable t defined on $[0, \infty)$. Sometimes we shall suppose that $f(t)$ is defined on $(-\infty, \infty)$, but for $t < 0$ the function $f(t) \equiv 0$. In addition, we shall assume that the function $f(t)$ is piecewise continuous and on each finite interval has a finite number of points of discontinuity of the first kind. Let $p = a + ib$ be a complex number.

Consider the function

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt. \tag{1}$$

If

$$|f(t)| \leqslant M \exp(s_0 t), \tag{2}$$

where $a > s_0$, then the function $F(p)$ is analytic in the half-plane $\operatorname{Re} p > s_0$ (Fig. 154).

Indeed,

$$\begin{aligned} |F'(p)| &= \left| \int_0^\infty t \exp(-at - ibt) f(t)\, dt \right| \leqslant \int_0^\infty t \exp(-at) |f(t)|\, dt \\ &\leqslant \int_0^\infty \exp(-at) \cdot tM \exp(s_0 t)\, dt \\ &= M \int_0^\infty t \exp(-(a - s_0) t)\, dt < \infty, \end{aligned} \tag{3}$$

since $a > s_0$. The validity of differentiation with respect to p under the integral follows from inequality (3) and the

fact that the function $tf(t) \exp(-pt)$ is piecewise continuous (see Sec. 2.15, Theorem 2).

The function $F(p)$ is called the *Laplace transform* (*L-transform*) or the *Laplace transformation*.

We shall use the following notation:

$$F(p) = L(f(t);\ p),\quad f(t) \doteq F(p),\quad F(p) \doteq f(t).$$

In this case the function $f(t)$ is termed the *original function* or the *object*. The number s_0 ($s_0 = s_0(f)$) is called the

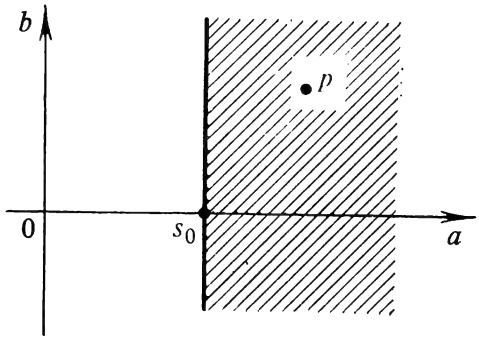

Fig. 154.

index of the order of growth of the function $f(t)$ (below, if otherwise is not stated, we assume the index of the order of growth of f to be equal to s_0).

The process of finding the transform for a given original function and, conversely, finding the original function by a given transform is called the *operational calculus* which was invented by O. Heaviside*. Among other things, he considered the transform

$$\widetilde{F}(p) = p\int_0^\infty \exp(-pt)\, f(t)\, dt,$$

i.e.

$$\widetilde{F}(p) = pF(p).$$

* Heaviside, Oliver (1850-1925). English electrical engineer. His methods of studying operators were developed experimentally rather than logically, but many of them have been rigorously justified by other investigators.

In certain cases the Laplace transform is more convenient, in others the Heaviside transform. We shall consider the Laplace transform.

Operational calculus was rigorously justified by a number of mathematicians in 20's of our century.

Theorem 1 (of uniqueness). *If two continuous functions $f(t)$ and $g(t)$ have one and the same L-transform $F(p)$, then they are identically equal.*

The proof of this theorem we leave to the reader.

On the basis of Theorem 1, we can say that for a continuous function $f(t)$, identically not equal to zero, the transform cannot be a periodic function.

Indeed, if $\forall p\ F(p) = F(p + \omega)$, where $\omega \neq 0$, then

$$\int_0^\infty \exp(-pt) f(t)\, dt = \int_0^\infty \exp(-(p+\omega)t) f(t)\, dt.$$

By Theorem 1,

$$f(t) = \exp(-\omega t) f(t),$$

that is, $\exp(-\omega t) \equiv 1$ $(\omega \neq 0)$, which is impossible.

Sec. 7.2. Transforms of Elementary Functions and Their Properties

The function

$$\sigma_0(t) = \begin{cases} 1, & t \geqslant 0, \\ 0, & t < 0 \end{cases}$$

is called the *Heaviside unit function*. Obviously, the index of the order of growth of this function $s_0 = 0$. Let us find the L-transform of this function in the domain $\operatorname{Re} p > 0$:

$$L(\sigma_0(t); p) = \int_0^\infty \exp(-pt)\, dt = -\frac{1}{p} \exp(-pt)\Big|_0^\infty = \frac{1}{p}.$$

Thus,

$$\sigma_0(t) \doteq \frac{1}{p}. \tag{1}$$

Analogously, integrating by parts, we find for the function $f(t) = \cos t$:

$$L(\cos t;\ p) = \int_0^\infty \exp(-pt)\cos t\,dt = \{\exp(-pt) = u,\ \cos t\,dt = dv\}$$

$$= \exp(-pt)\sin t\Big|_0^\infty + p\int_0^\infty \exp(-pt)\sin t\,dt$$

$$= p\int_0^\infty \exp(-pt)\sin t\,dt = p\left[-\exp(-pt)\cos t\Big|_0^\infty - p\int_0^\infty \exp(-pt)\cos t\,dt\right] = p - p^2 L[\cos t;\ p].$$

Hence,

$$L(\cos t;\ p) = \frac{p}{1+p^2} \quad (\operatorname{Re} p > 0),$$

i. e.

$$\cos t \doteqdot \frac{p}{1+p^2}. \tag{2}$$

Simultaneously we have proved that

$$L(\cos t;\ p) = pL(\sin t;\ p),$$

whence

$$\sin t \doteqdot \frac{1}{1+p^2}. \tag{3}$$

Theorem 1 (similarity theorem). *If* $f(t) \doteqdot F(p)$, *then* $f(\alpha t) \doteqdot \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right)$ $(\alpha > 0,\ \operatorname{Re} p > \max\{s_0, \alpha s_0\})$.

Indeed,

$$L[f(\alpha t);\ p] = \int_0^\infty \exp(-pt) f(\alpha t)\,dt = \left\{\alpha t = u,\ dt = \frac{du}{\alpha}\right\}$$

$$= \frac{1}{\alpha}\int_0^\infty \exp\left(-\frac{p}{\alpha}u\right) f(u)\,du = \frac{1}{\alpha} L\left[f(u);\ \frac{p}{\alpha}\right].$$

On the basis of Theorem 1, we obtain

$$\cos \alpha t \doteqdot \frac{1}{\alpha} \frac{p/\alpha}{1+(p/\alpha)^2} = \frac{p}{p^2+\alpha^2}; \tag{4}$$

$$\sin \alpha t \doteqdot \frac{1}{\alpha} \frac{1}{1+(p/\alpha)^2} = \frac{\alpha}{p^2+\alpha^2}. \tag{5}$$

Theorem 2 (linearity property). *There takes place the equality*

$$L[Af(t) + Bg(t); p] = AL[f(t); p] + BL[g(t); p]$$

where A, B are constant numbers.

This property follows from the corresponding property of improper integral. Note that if the indices of the order of growth of the functions f and g are respectively equal to s_0 and $\bar{s}_0$, then the transform $Af + Bg$ exists in the half-plane $\operatorname{Re} p > \max \{s_0, \bar{s}_0\}$.

Example 1. Find the transform of the function

$$f(t) = 3\sigma_0(t) + 2\cos 3t.$$

By (1), (4), and Theorem 2, we have

$$L[f(t); p] = 3L[\sigma_0(t); p] + 2L[\cos 3t; p] = \frac{3}{p} + 2\frac{p}{p^2+9}.$$

Example 2. Find the original function of the transform

$$F(p) = \frac{2}{p} + \frac{2}{p^2+16}.$$

Let us represent the transform $F(p)$ in the form

$$F(p) = 2 \cdot \frac{1}{p} + \frac{1}{2}\frac{4}{p^2+4^2}.$$

We have

$$\frac{1}{p} \doteqdot \sigma_0(t), \quad \frac{4}{p^2+4^2} \doteqdot \sin 4t.$$

Consequently, the object (by Theorem 1, Sec. 7.1)

$$f(t) = 2\sigma_0(t) + \frac{1}{2}\sin 4t.$$

Theorem 3 (shift of image function)

$L[f(t) \exp(-\alpha t); p] = L[f(t); p + \alpha]$, $\operatorname{Re}(p + \alpha) > s_0$.

The proof is obvious.

Example 3. Find the transforms of the functions $e^{-\alpha t}\cos\beta t$, $e^{-\alpha t}\sin\beta t$, $e^{-\alpha t}$.

Since

$$\cos\beta t \doteqdot \frac{p}{p^2+\beta^2},$$

then, by Theorem 3

$$L[\exp(-\alpha t)\cos\beta t;\ p] = \frac{p+\alpha}{(p+\alpha)^2+\beta^2}. \tag{6}$$

Just in a similar way, using formulas (5) and (1), we have

$$L[\exp(-\alpha t)\sin\beta t;\ p] = \frac{\beta}{(p+\alpha)^2+\beta^2}, \tag{7}$$

$$L[\exp(-\alpha t);\ p] = L[\sigma_0(t);\ p+\alpha] = \frac{1}{p+\alpha}. \tag{8}$$

Example 4. Find $L[\cosh\alpha t;\ p]$, $L[\sinh\alpha t;\ p]$.

Using Theorem 2 and equality (8), we have

$$L[\cosh\alpha t;\ p] = L\left[\frac{e^{\alpha t}+e^{-\alpha t}}{2};\ p\right] = \frac{1}{2}L[e^{\alpha t};\ p] + \frac{1}{2}L[e^{-\alpha t};\ p]$$

$$= \frac{1}{2}\frac{1}{p-\alpha} + \frac{1}{2}\frac{1}{p+\alpha} = \frac{p}{p^2-\alpha^2}.$$

Analogously,

$$L[\sinh\alpha t;\ p] = \frac{\alpha}{p^2-\alpha^2}.$$

Example 5. Find the original function for the transform $F(p) = 1/(p^2 + 2p + 5)$.

We have

$$F(p) = \frac{2}{2[(p+1)^2+2^2]},\quad \frac{2}{p^2+2^2} \doteqdot \sin 2t,$$

$$\frac{2}{(p+1)^2+2^2} \doteqdot e^{-t}\sin 2t,\quad f(t) = \frac{1}{2}e^{-t}\sin 2t.$$

Theorem 4 (differentiation of image function).

$$(-1)^n \frac{d^n}{dp^n} L[f(t);\ p] = L[t^n f(t);\ p].$$

Proof. If $\operatorname{Re} p > s_0$, where s_0 is the index of the order of growth of the function $f(t)$, then the integral

$$\int_0^\infty t^n f(t) \exp(-pt)\, dt$$

exists for any $n = 1, 2, \ldots$. Further, it is obvious that

$$\frac{d^n}{dp^n} \int_0^\infty \exp(-pt) f(t)\, dt = \int_0^\infty (-t)^n \exp(-pt) f(t)\, dt.$$

Hence,

$$(-1)^n L[f(t) t^n;\ p] = \frac{d^n}{dp^n} L[f(t);\ p].$$

Example 6. Since $\frac{1}{p} \doteqdot \sigma_0(t)$, by Theorem 4, we obtain

$$(-1) \frac{d}{dp}\left(\frac{1}{p}\right) \doteqdot 1 \cdot t,$$

i. e.

$$t \doteqdot \frac{1}{p^2}.$$

Continuing differentiation, we get

$$t^n \doteqdot \frac{n!}{p^{n+1}} \quad (n = 1, 2, \ldots). \tag{9}$$

If n is not integral, then

$$t^n \doteqdot \frac{\Gamma(n+1)}{p^{n+1}},$$

where

$$\Gamma(a+1) = \int_0^\infty e^{-t} t^a\, dt = L[t^a;\ 1].$$

For a natural n we have $\Gamma(n+1) = n!$.

Example 7. Find the transform of the function $t \cos \alpha t$.

We have

$$\frac{p}{p^2+\alpha^2} \doteqdot \cos \alpha t, \quad -\left(\frac{p}{p^2+\alpha^2}\right)' \doteqdot t \cos \alpha t$$

or

$$\frac{p^2-\alpha^2}{(p^2+\alpha^2)^2} \doteq t\cos\alpha t. \tag{10}$$

Theorem 5 (differentiation of original function). *The formula*

$$L[f'(t);\ p] = pL[f(t);\ p] - f(0) \quad (\mathrm{Re}\ p > s_0) \tag{11}$$

holds true in the supposition that the function $f(t)$ is continuous, has a piecewise continuous derivative $f'(t)$ on $[0, \infty]$ with discontinuities of the first kind and the indices of the order of growth of $f(t)$ and $f'(t)$ are equal to s_0.*

Proof. We have

$$L[f'(t);\ p] = \int_0^\infty \exp(-pt)\, f'(t)\, dt = \lim_{N\to\infty} \int_0^N \exp(-pt)\, f'(t)\, dt$$

$$= \lim_{N\to\infty} \left[e^{-pt} f(t) \Big|_0^N + p \int_0^N e^{-pt} f(t)\, dt \right]$$

$$= -f(0) + pL[f(t);\ p] \quad (\mathrm{Re}\, p > s_0),$$

since

$$|e^{-pN} f(N)| \leqslant e^{-\alpha N} M e^{s_0 N} = M e^{-(\alpha - s_0)N} \xrightarrow[N\to\infty]{} 0.$$

Corollary 1. *The following formula holds true*

$$L[f^{(n)}(t);\ p] = p^n L[f(t);\ p] - p^{n-1} f(0) - \ldots - p f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$(\mathrm{Re}\ p > s_0) \tag{12}$$

on the condition that $f(t), \ldots, f^{(n-1)}(t)$ are continuous, $f^{(n)}$ is piecewise continuous on $[0, \infty]$, and the growth of the function f together with its derivatives to order n included is equal to s_0.

---

* There are cases when the function $f(t)$ mentioned in the theorem is defined on the interval $(0, \infty)$ and there exists the right-hand limit $f(0+0) = \lim\limits_{\substack{t\to 0 \\ t>0}} f(t)$. Then in formula (11) $f(0)$ should be replaced by $f(0+0)$.

In particular, for

$$f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0 \tag{13}$$

there takes place

$$L[f^{(n)}(t);\ p] = p^n L[f(t);\ p]. \tag{14}$$

Example 8. Find the transform of the function $f(t) = \cos^2 t$.

Let $F(p) \doteqdot \cos^2 t = f(t)$. Then

$$f'(t) \doteqdot pF(p) - f(0).$$

But

$$f(0) = \cos^2 0 = 1, \quad f'(t) = -2 \sin t \cos t = -\sin 2t \doteqdot -\frac{2}{p^2+4}.$$

Consequently,

$$pF(p) - 1 = -\frac{2}{p^2+4},$$

whence

$$F(p) = \frac{1}{p}\left[1 - \frac{2}{p^2+4}\right] = \frac{p^2+2}{p(p^2+4)}.$$

The same result can be obtained by using the equality

$$\cos^2 t = \frac{1}{2} + \frac{\cos 2t}{2}, \quad L[\cos^2 t;\ p] = \frac{1}{2p} + \frac{1}{2}\,\frac{p}{p^2+4} = \frac{p^2+2}{p(p^2+4)}.$$

Theorem 6 (integration of original function).

$$\int_0^t f(\tau)\,d\tau \doteqdot \frac{F(p)}{p}.$$

Indeed, changing the order of integration, we have

$$\int_0^\infty e^{-pt} \int_0^t f(\tau)\,d\tau\,dt = \int_0^\infty \int_\tau^\infty e^{-pt} f(\tau)\,d\tau\,dt$$

$$= \int_0^\infty f(\tau)\,\frac{-e^{-pt}}{p}\Big|_\tau^\infty d\tau = \frac{1}{p}\int_0^\infty e^{-p\tau} f(\tau)\,d\tau = \frac{1}{p}\,L[f(\tau);\ p] = \frac{F(p)}{p}.$$

Theorem 7 (integration of image function). *If the integral* $\int_p^\infty F(q)\,dq$ *converges, then it is the transform of the function* $f(t)/t$:

$$\frac{f(t)}{t} \doteqdot \int_p^\infty F(q)\,dq.$$

Proof. Changing the order of integration, we obtain

$$\int_p^\infty F(q)\,dq = \int_p^\infty \left(\int_0^\infty e^{-qt} f(t)\,dt \right) dq$$

$$= \int_0^\infty \left(\int_p^\infty e^{-qt}\,dq \right) f(t)\,dt = \int_0^\infty \left(-\frac{e^{-qt}}{t}\Big|_{q=p}^{\infty} \right) f(t)\,dt$$

$$= \int_0^\infty \frac{f(t)}{t} e^{-pt}\,dt = L\left(\frac{f(t)}{t}; p\right).$$

Remark 1. Here and below we apply the change of the order of integration. According to Fubini's theorem, which is not proved here, this operation is valid if the multiple integral obtained after the change is absolutely convergent. Besides, we suppose that the path of integration lies in the half-plane Re $p > s_0$.

Corollary 2.

$$\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(q)\,dq,$$

if the corresponding improper integrals are convergent.

Example 9. Find the transform of the function $\int_0^t \sin 2\tau\,d\tau$.

We have

$$\sin 2\tau \doteqdot \frac{2}{p^2+4}.$$

By Theorem 6,

$$\int_0^t \sin 2\tau \, d\tau \doteq \frac{2}{p(p^2+4)}.$$

Example 10. Find the transform of the function $\frac{\sin t}{t}$. We know that $\sin t \doteq \frac{1}{p^2+1}$. Therefore, by Theorem 7,

$$\frac{\sin t}{t} \doteq \int_p^\infty \frac{dq}{q^2+1} = \operatorname{arc\,tan} q \Big|_p^\infty = \frac{\pi}{2} - \operatorname{arc\,tan} p.$$

Example 11. Find the integral

$$\int_0^\infty \frac{\sin t}{t} \, dt.$$

Using Example 10 and Corollary 2, we obtain

$$\int_0^\infty \frac{\sin t}{t} \, dt = \int_0^\infty \frac{dq}{q^2+1} = \operatorname{arc\,tan} q \Big|_0^\infty = \frac{\pi}{2}.$$

Theorem 8 (delay of original function). *Let* $f(t) = 0$ *for* $t < 0$, *then*

$$L[f(t-t_0);\ p] = e^{-pt_0} L[f(t);\ p],$$

where t_0 *is some point.*

Proof. We have

$$L[f(t-t_0);\ p] = \int_0^\infty e^{-pt} f(t-t_0)\, dt$$

$$= \int_0^{t_0} e^{-pt} f(t-t_0)\, dt + \int_{t_0}^\infty e^{-pt} f(t-t_0)\, dt$$

$$= \int_{t_0}^\infty e^{-pt} f(t-t_0)\, dt = \{t - t_0 = u,\ dt = du\}$$

$$= \int_0^\infty e^{-p(u+t_0)} f(u)\, du = e^{-pt_0} \int_0^\infty e^{-pu} f(u)\, du = e^{-pt_0} L[f(u);\ p].$$

Example 12. Since $L[\sigma_0(t);\ p] = \frac{1}{p}$, then (Fig. 155)

$$L(\sigma_0(t-h);\ p] = e^{-ph}\frac{1}{p}.$$

Example 13. Let (Fig. 156)

$$f(t) = \sigma_0(t-h) - \sigma_0(t-h_1) \quad (h < h_1).$$

By Theorems 2 and 8, we have

$$L[f(t);\ p] = L[\sigma_0(t-h);\ p] - L[\sigma_0(t-h_1);\ p] = \frac{e^{-ph} - e^{-ph_1}}{p}.$$

Example 14. Find the transform of the function $f(t)$

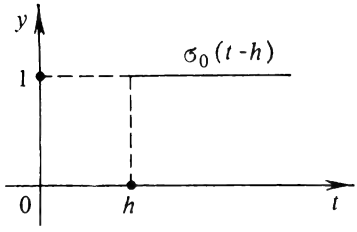

Fig. 155.

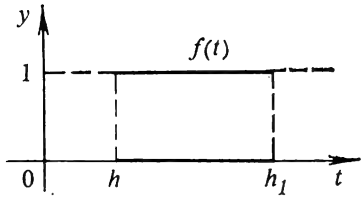

Fig. 156.

(Fig. 157) defined on the closed interval $[0, 2a]$ by the equalities

$$f(t) = \begin{cases} A, & 0 \leqslant t \leqslant a, \\ 0 & a < t < 2a \end{cases}$$

and extended then to the entire ray to $t > 0$ with period $2a$.

We have

$$L[f(t);\ p] = \int_0^\infty e^{-pt} f(t)\,dt = \sum_{k=0}^{\infty} \int_{ka}^{(k+1)a} e^{-pt} f(t)\,dt$$

$$= \sum_{k=0}^{\infty} \int_{2ka}^{(2k+1)a} e^{-pt} A\,dt = \sum_{k=0}^{\infty} \frac{A}{p}\left[e^{-p2ka} - e^{-p(2k+1)a}\right]$$

$$= \frac{A(1-e^{-pa})}{p} \sum_{k=0}^{\infty} e^{-2kpa} = \frac{A(1-e^{-pa})}{p(1-e^{-2pa})} = \frac{A}{p(1+e^{-pa})}.$$

The expression

$$\int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau$$

is called the *convolution of the functions* $f_1(t)$ and $f_2(t)$ and is denoted by the symbol $f_1 * f_2$.

It is readily checked that

$$\int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau = \int_0^t f_1(t-\tau)\, f_2(\tau)\, d\tau$$

(by changing the variable: $t - \tau = u$).

Theorem 9. *The Laplace transform of a convolution is equal*

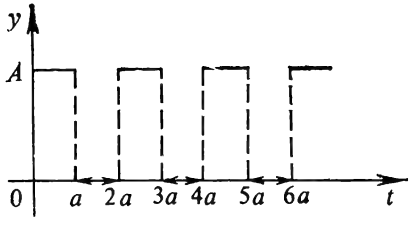

Fig. 157.

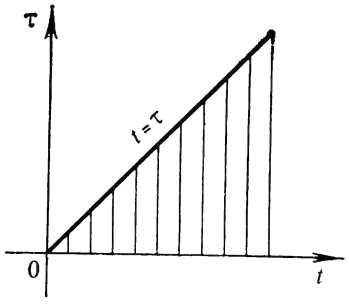

Fig. 158.

to the product of the Laplace transform of the functions $f_1(t)$ *and* $f_2(t)$ $(s_0(f_1) = s_0(f_2))$:

$$L\Big[\int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau;\ p\Big] = L[f_1(t);\ p]\cdot L[f_2(t);\ p].$$

Proof. Let us recall that we assume that $f_1(t) = f_2(t) = 0$ for $t < 0$. Changing the order of integration (see Fig. 158) and taking into consideration that $f_2(t-\tau) = 0$ for $0 < t < \tau$, we have

$$L[f_1 * f_2;\ p] = \int_0^\infty e^{-pt} \int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau\, dt$$

$$= \int_0^\infty \int_\tau^\infty e^{-pt} f_1(\tau)\, f_2(t-\tau)\, dt\, d\tau$$

$$= \int_0^\infty f_1(\tau) \left(\int_\tau^\infty e^{-pt} f_2(t-\tau)\, dt \right) d\tau = \{t-\tau = z,\ dt = dz\}$$

$$= \int_0^\infty f_1(\tau) \left(\int_0^\infty e^{-p(z+\tau)} f_2(z)\, dz \right) d\tau$$

$$= \int_0^\infty f_1(\tau)\, e^{-p\tau}\, d\tau \int_0^\infty f_2(z)\, e^{-pz}\, dz = L[f_1(\tau);\ p] \cdot L[f_2(z);\ p].$$

Notice that the double integral of the function $e^{-pt} f_1(\tau) f_2(t-\tau)$ over the infinite sector $\{0 < \tau \leqslant t,\ 0 < t < \infty\}$ is absolutely convergent for $\operatorname{Re} p > s_0$.

Example 15.

$$L\left[\int_0^t e^{t-\tau} \cosh \alpha\tau\, d\tau;\ p \right] = L[e^t;\ p]\, L[\cosh \alpha t;\ p]$$

$$= \frac{1}{p-1} \cdot \frac{p}{p^2-\alpha^2}.$$

Corollary 3. *Let* $F(p) \doteqdot f(t)$, $G(p) \doteqdot g(t)$, *then the Duhamel* formula takes place*

$$pF(p)\, G(p) \doteqdot f(t)\, g(0) + \int_0^t f(\tau)\, g'(t-\tau)\, d\tau. \qquad (15)$$

Proof. We have

$$F(p)\, G(p) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Hence, by Theorem 5 on differentiation of original function, we get

$$pF(p)\, G(p) \doteqdot \left(\int_0^t f(\tau)\, g(t-\tau)\, d\tau \right)'_t = f(t)\, g(0) + \int_0^t f(\tau)\, g'(t-\tau)\, d\tau.$$

* Duhamel, Jean Marie Constant (1797-1872). French analyst and applied mathematician.

See also the footnote on p. 454.

Theorem 10. *If* $\mathcal{F}(f(x); y)$ *and* $L[f(x); p]$ *are the Fourier transform and the Laplace transform of the function* f, *respectively,* $(s_0(f) < 0)$, *then*

$$2\pi\mathcal{F}(f(x); y) = L[f(x); iy] + L[f(-x); -iy]. \quad (16)$$

Indeed,

$$2\pi\mathcal{F}(f(x); y) = \int_{-\infty}^{\infty} e^{-ixy} f(x)\,dx = \int_{0}^{\infty} e^{-ixy} f(x)\,dx + \int_{-\infty}^{\infty} e^{-ixy} f(x)\,dx$$

$$= L[f(x); iy] + \int_{0}^{\infty} e^{ixy} f(-x)\,dx = L[f(x); iy] + L[f(-x); -iy].$$

If the Laplace transform of the function f is known, then the Fourier transform is readily found by formula (16).

Example 16. Let

$$f(x) = \begin{cases} e^{-\alpha x}\cos\beta x, & x \geqslant 0, \quad \alpha > 0, \\ 0, & x < 0. \end{cases}$$

Find the Fourier transform of this function.

The Laplace transform of the function f exists $(s_0(f) = -\alpha)$, therefore

$$2\pi\mathcal{F}(f(x); y) = L[f(x); iy] = L[e^{-\alpha x}\cos\beta x; iy] = \frac{iy+\alpha}{(iy+\alpha)^2+\beta^2}.$$

Stated below is a number of theorems on finding the original function given its transform. (Their proof is left to the reader.)

Theorem 11. *Let* $F(p)$ *be an analytic function on the extended complex plane and let* $p = \infty$ *be a regular point and* $F(\infty) = 0$, *i.e. the Laurent series has the form*

$$F(p) = \sum_{k=1}^{\infty} \frac{c_k}{p^k}.$$

Then the original function of this transform is given by the formula

$$f(t) = \begin{cases} 0, & t < 0, \\ \sum\limits_{n=0}^{\infty} c_{n+1}\dfrac{t^n}{n!}, & t > 0. \end{cases} \quad (17)$$

Indeed,

$$\int_0^\infty f(t)\, e^{-pt}\, dt = \sum_{n=0}^{\infty} \frac{c_{n+1}}{n!} \int_0^\infty e^{-pt} t^n\, dt = \sum_{n=0}^{\infty} \frac{c_{n+1}}{p^{n+1}} = \sum_{k=1}^{\infty} \frac{c_k}{p^k}.$$

By virtue of Theorem 1 (uniqueness theorem) from the preceding section, the theorem has been proved.

Theorem 12. *Let $F(p)$ be a rational fractional function with the poles $p_1, p_2, \ldots, p_m$. Then*

$$f(t) = \sum_{k=1}^{m} \operatorname*{Res}_{p=p_k} [F(p)\, e^{pt}]. \tag{18}$$

If p_k are simple poles and $F(p) = A(p)/B(p)$, where $A(p)$, $B(p)$ are polynomials having no common roots, then

$$f(t) = \sum_{k=1}^{m} \frac{A(p_k)}{B'(p_k)}\, e^{p_k t}. \tag{19}$$

Theorem 13 (Mellin's* formula). *If $F(p)$ is an analytic function in* $\operatorname{Re} p > s_0$, $F(p) \to 0$ *uniformly with respect to* $\arg p$, *as* $|p| \to \infty$, $\int_{x-i\infty}^{x+i\infty} |F(p)|\, dy < M$, *then $F(p)$ is the transform of the function*

$$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{pt} F(p)\, dp \quad (x > s_0). \tag{20}$$

Example 17. Find the original of the function

$$F(p) = 1/(p-1)(p^2+1).$$

We shall take advantage of Theorem 12. Here $A(p) \equiv 1$, $B(p) = (p-1)(p^2+1)$. The points $p = 1$, $p = \pm i$ are the simple poles of the function $F(p)$. By formula (19),

* Mellin, Robert Hjalmar (1854-1933). Finnish analyst and mathematical physicist.

we have $(B'(p) = 3p^2 - 2p + 1)$

$$f(t) = \frac{e^t}{2} - \frac{e^{it}}{2(1+i)} + \frac{e^{-it}}{2(i-1)} = \frac{1}{2}[e^t - \cos t - \sin t].$$

Example 18. Find the original function $f(t)$ if $F(p) = \sin(1/p)$.

We have

$$\sin\frac{1}{p} = \frac{1}{p} - \frac{1}{3!\,p^3} + \frac{1}{5!\,p^5} - \cdots,$$

i.e. $F(p)$ satisfies the conditions of Theorem 11. Therefore

$$f(t) = 1 - \frac{1}{3!}\frac{t^2}{2!} + \frac{1}{5!}\frac{t^4}{4!} - \cdots.$$

For the sake of convenience, let us tabulate all the obtained transforms of elementary functions.

No.	Original function	Transform
1	1	$\frac{1}{p}$
2	$\sin \alpha t$	$\frac{\alpha}{p^2+\alpha^2}$
3	$\cos \alpha t$	$\frac{p}{p^2+\alpha^2}$
4	$\cos \alpha (t-t_0)$	$\frac{pe^{-pt_0}}{p^2+\alpha^2}$
5	$e^{-\alpha t}$	$\frac{1}{p+\alpha}$
6	$\sinh \alpha t$	$\frac{\alpha}{p^2-\alpha^2}$
7	$\cosh \alpha t$	$\frac{p}{p^2-\alpha^2}$
8	$e^{-\alpha t} \sin \beta t$	$\frac{\beta}{(p+\alpha)^2+\beta^2}$
9	$e^{-\alpha t} \cos \beta t$	$\frac{p+\alpha}{(p+\alpha)^2+\beta^2}$

No.	Original function	Transform
10	t^n	$\dfrac{\Gamma(n+1)}{p^{n+1}}$
11	$t^n f(t)$	$(-1)^n \dfrac{d^n F(p)}{dp^n}$
12	$te^{-\alpha t}$	$\dfrac{1}{(p+\alpha)^2}$
13	$t \sin \alpha t$	$\dfrac{2p\alpha}{(p^2+\alpha^2)^2}$
14	$t \cos \alpha t$	$\dfrac{p^2-\alpha^2}{(p^2+\alpha^2)^2}$
15	$f^{(n)}(t),\ f(0) = \ldots = f^{(n-1)}(0) = 0$	$p^n F(p)$
16	$\int\limits_0^t f(\tau)\, d\tau$	$\dfrac{1}{p} F(p)$
17	$\dfrac{f(t)}{t}$	$\int\limits_p^\infty F(q)\, dq$
18	$f(t-t_0)$	$e^{-pt_0} F(p)$
19	$\sigma_0(t-h)$	$e^{-ph} \dfrac{1}{p}$
20	$f_1 * f_2$	$L[f_1;\ p] \cdot L[f_2;\ p]$
21	$f(t)\, g(0) + \int\limits_0^t f(\tau)\, g'(t-\tau)\, d\tau$	$pL[f;\ p] \cdot L[g;\ p]$
22	$\sum\limits_{k=0}^{\infty} c_{k+1} \dfrac{t^k}{k!}\ (t > 0)$	$\sum\limits_{k=1}^{\infty} \dfrac{c_k}{p^k}$

Sec. 7.3. Applications of Operational Calculus

Let there be given a linear differential equation of order n with constant coefficients

$$a_n x^{(n)}(t) + \ldots + a_1 x'(t) + a_0 x(t) = f(t). \qquad (1)$$

It is required to find the solution of equation (1) for $t \geqslant 0$ at the initial conditions:

$$x(0) = x_0, \quad x'(0) = x_0', \quad \ldots, \quad x^{(n-1)}(0) = x_0^{(n-1)}. \qquad (2)$$

Let $x(t)$ be the solution of (1) satisfying the initial conditions (2). Then, on substituting this function into (1), we obtain an identity. Hence, the function standing on the left-hand side of (1) and the function $f(t)$ have one and the same L-transform:

$$L\left[\sum_{k=0}^{n} a_k \frac{d^k x}{dt^k};\ p\right] = L[f(t);\ p].$$

By Corollary 1 of Sec. 7.2

$$L\left[\frac{d^k x}{dt^k};\ p\right] = p^k L[x;\ p] - p^{k-1}x(0) - \ldots - px^{(k-2)}(0) - x^{(k-1)}(0).$$

Therefore, using the linearity property of the transform, we get

$$a_n L\left[\frac{d^n x}{dt^n};\ p\right] + \ldots + a_0 L[x;\ p] = L[f;\ p];$$

$$a_n [p^n L[x;\ p] - p^{n-1}x_0 - p^{n-2}x_0' - \ldots - px_0^{(n-2)} - x_0^{(n-1)}$$
$$+ a_{n-1}[p^{n-1}L[x;\ p] - p^{n-2}x_0 - \ldots - px_0^{n-3} - x_0^{(n-2)}]$$
$$+ \ldots + a_1 [L[x;\ p] - x_0] + a_0 L[x;\ p] = L[f;\ p].$$

For the sake of brevity of the notation, let us set $L[x;\ p] = \bar{x}(p)$, $L[f;\ p] = F(p)$. Then

$$\bar{x}(p) \cdot [a_n p^n + a_{n-1}p^{n-1} + \ldots + a_1 p + a_0]$$
$$= a_n [p^{n-1}x_0 + p^{n-2}x_0' + \ldots + x_0^{(n-1)}]$$
$$+ a_{n-1}[p^{n-2}x_0 + p^{n-3}x_0' + \ldots + x_0^{(n-2)}]$$
$$+ \ldots + a_2 [px_0 + x_0'] + a_1 x_0 + F(p). \qquad (3)$$

Equation (3) will be called an *auxiliary equation or transforming equation*, or *operator equation*.

Note that the coefficient of $\bar{x}(p)$ in (3) is obtained from the left-hand side of (1) by formally replacing the derivatives $\frac{d^k x}{dt^k}$ by powers of p^k. Let us denote this coefficient by

$$R_n(p) = a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_0.$$

It is easily seen that this coefficient is the left-hand side of the characteristic equation for differential equation (1) (see Sec. 1.16, (2)). Then the transform of the solution is found in the form

$$\bar{x}(p) = \frac{F(p)}{R_n(p)} + \frac{\psi_{n-1}(p)}{R_n(p)}, \tag{4}$$

where

$$\psi_{n-1}(p) = a_1 x_0 + a_2 (p x_0 + x_0') + a_3 (p^2 x_0 + p x_0' + x_0'')$$
$$+ \ldots + a_n [p^{n-1} x_0 + p^{n-2} x_0' + \ldots + p x_0^{(n-2)} + x_0^{(n-1)}].$$

If the initial conditions are zero, i.e. $x_0 = \ldots = x_0^{(n-1)} = 0$, then formula (4) will be written as

$$\bar{x}(p) = \frac{F(p)}{R_n(p)}. \tag{4'}$$

If now, knowing transform (4) or (4') we find the original, then, by virtue of the uniqueness theorem, this will be the required solution $x(t)$.

Example 1. Solve the equation

$$\ddot{x} + 4x = 2, \quad x_0 = x_0' = 0.$$

By formula (4'), we have

$$\bar{x}(p) = \frac{2}{p(p^2+4)},$$

since $2 \doteq 2/p$. Let us expand the transform into partial fractions

$$\bar{x}(p) = \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right].$$

Hence,

$$x(t) = \frac{1}{2}\sigma_0(t) - \frac{1}{2}\cos 2t = \frac{1}{2} - \frac{1}{2}\cos 2t.$$

We have obtained the solution $(1 - \cos 2t)/2$ only for $t \geqslant 0$. It is easy to check that it also satisfies our equation

for $t < 0$. However, this fact follows from the general considerations which are not dealt with here. This remark also refers to Examples 2-4.

We may also take advantage of Theorem 12 from the preceding section ($A \equiv 2$, $B = p(p^2 + 4)$, $B' = 3p^2 + 4$; 0, $\pm 2i$ are simple zeros of the polynomial $B(p)$):

$$x(t) = 2\left[\frac{e^{0t}}{B'(0)} + \frac{e^{2it}}{B'(2i)} + \frac{e^{-2it}}{B'(-2i)}\right]$$

$$= 2\left[\frac{1}{4} + \frac{1}{-8}e^{2it} + \frac{1}{-8}e^{-2it}\right] = \frac{1}{2} - \frac{1}{2}\cos 2t.$$

Example 2. $y'' + 2y' + 5y = \sin x$, $y(0) = 0$, $y'(0) = 1$.

Let us write the auxiliary equation:

$$[p^2\bar{y}(p) - py(0) - y'(0)] = 2[p\bar{y}(p) - y(0)] + 5\bar{y}(p) = \frac{1}{p^2+1},$$

$$\bar{y}(p)(p^2 + 2p + 5) = \frac{1}{p^2+1} + 1.$$

Hence,

$$\bar{y}(p) = \frac{1}{(p^2+1)(p^2+2p+5)} + \frac{1}{p^2+2p+5} = \frac{p^2+2}{(p^2+1)(p^2+2p+5)}.$$

The polynomial $B(p) = (p^2 + 1)(p^2 + 2p + 5)$ has the simple zeros $p = \pm i$, $p = -1 \pm 2i$. On the basis of Theorem 12 stated in the preceding section ($A = p^2 + 2$, $B'(p) = 2p(p^2 + 2p + 5) + 2(p + 1)(p^2 + 1)$), we have:

$$A(\pm i) = 1,\ B'(i) = 4i(2 + i),\ B'(-i) = -4i(2-i),$$

$$A(-1 + 2i) = -1 - 4i;\ B'(-1 + 2i) = -8i(2i+1),$$

$$A(-1 - 2i) = 4i - 1,\ B'(-1-2i) = -8i(2i - 1),$$

$$y(x) = \frac{e^{ix}}{4i(2+i)} + \frac{e^{-ix}}{-4i(2-i)} + \frac{-(1+4i)e^{(-1+2i)x}}{-8i(2i+1)}$$

$$+ \frac{(4i-1)e^{-(1+2i)x}}{-8i(2i-1)} = \frac{\sin x}{5} - \frac{\cos x}{10} + e^{-x}\left[\frac{\cos 2x}{10} + \frac{9}{20}\sin 2x\right].$$

It is also possible to use the method of expanding the transform into partial fractions:

$$\frac{p^2+2}{(p^2+1)(p^2+2p+5)} = \frac{Ap+B}{p^2+1} + \frac{Cp+D}{p^2+2p+5}.$$

The numbers A, B, C, D are found by the method of undetermined coefficients: $A = -1/10$, $B = 1/5$, $C = 1/10$, $D = 1$.

Thus,

$$\bar{y}(p) = \frac{-\frac{p}{10}+\frac{1}{5}}{p^2+1} + \frac{\frac{p}{10}+1}{p^2+2p+5} = -\frac{1}{10}\frac{p}{p^2+1} + \frac{1}{5}\frac{1}{p^2+1} + \frac{1}{10}\frac{p+1}{(p+1)^2+4} + \frac{9}{20}\frac{2}{(p+1)^2+4}.$$

According to the table of transforms, we have

$$y(x) = -\frac{1}{10}\cos x + \frac{1}{5}\sin x + \frac{1}{10}e^{-x}\cos 2x + \frac{9}{20}e^{-x}\sin 2x.$$

In solving a differential equation it is sometimes convenient to use the Duhamel formula (see Sec. 7.2, (15)).

We shall consider equation (1) at zero initial conditions: $x(0) = \ldots = x^{(n-1)}(0) = 0$. The problem can always be reduced to this case by substituting the required function by formula

$$x(t) = y(t) + \sum_{k=0}^{n-1} \frac{t^k}{k!} x^{(k)}(0).$$

Suppose the solution of equation (1) is known for the right-hand side equal to unity and zero initial conditions. The operator equation for the given problem has the form

$$R_n(p)\bar{x}_1(p) = \frac{1}{p}, \tag{5}$$

where $\bar{x}_1(p)$ is the transform of the solution $x_1(t)$ of the indicated problem. From equality (4′) and equation (5) we find

$$\bar{x}(p) = \frac{F(p)}{R_n(p)} = p\bar{x}_1(p)F(p). \tag{6}$$

According to the Duhamel formula

$$pF(p)\bar{x}_1(p) \doteq f(t)x_1(0) + \int_0^t f(\tau)x_1'(t-\tau)\,d\tau$$

or taking into account that $x_1(0) = 0$, we obtain

$$\bar{x}(p) = pF(p)\,\bar{x}_1(p) \doteq \int_0^t f(\tau)\,x_1'(t-\tau)\,d\tau.$$

Hence, the solution of equation (1) at zero initial conditions will have the form

$$x(t) = \int_0^t f(\tau)\,x_1'(t-\tau)\,dt, \tag{7}$$

where $x_1(t)$ is the solution of equation (1) for $f(t) \equiv 1$ and zero initial conditions.

Example 3. Solve the equation

$$x'' - x = \frac{1}{1+e^t}, \quad x(0) = x'(0) = 0.$$

Let us first solve the Cauchy problem for the equation

$$x_1'' - x_1 = 1, \quad x_1(0) = x_1'(0) = 0,$$

and then write the operator equation:

$$p^2\bar{x}_1(p) - \bar{x}_1(p) = \frac{1}{p}, \quad \bar{x}_1(p) = \frac{1}{p(p^2-1)} = \frac{p}{p^2-1} - \frac{1}{p}.$$

Hence,

$$x_1(t) = \cosh t - 1.$$

Remark. Since the right-hand side of the equation $x_1'' - x_1' = 1$ has a special form, the solution of this equation can be carried out in a usual way (see Sec. 1.18).

By formula (7),

$$x(t) = \int_0^t \frac{1}{1+e^\tau} \sinh(t-\tau)\,d\tau = \int_0^t \frac{e^{t-\tau} - e^{-t+\tau}}{2(1+e^\tau)}\,d\tau$$

$$= \frac{e^t}{2}\int_0^t \frac{e^{-\tau}d\tau}{1+e^\tau} - \frac{e^{-t}}{2}\int_0^t \frac{d(e^\tau+1)}{1+e^\tau}$$

$$= -\frac{e^{-t}}{2}\ln\frac{e^t+2}{2} - \frac{e^t}{2}\int_0^t \frac{e^{-\tau}\,de^{-\tau}}{e^{-\tau}+1}$$

$$= -\frac{e^{-t}}{2}\ln\frac{e^t+1}{2} - \frac{e^t}{2}(e^t-1) + \frac{e^t}{2}\int_0^t \frac{d(e^{-\tau}+1)}{e^{-\tau}+1}$$

$$= -\frac{e^{-t}}{2}\ln\frac{e^t+1}{2} - \frac{1}{2} + \frac{e^t}{2} + \frac{e^t}{2}\ln\frac{e^{-t}+1}{2}$$

$$= \sinh t \ln\frac{e^t+1}{2} + \frac{1}{2}[-te^t + e^t - 1].$$

Solving Systems of Differential Equations. Let us consider this question by studying a particular example.

Example 4. Find the solution of the linear system

$$\left.\begin{aligned} 2\dot{x} + \dot{y} + x &= 1, \\ \dot{x} + 3\dot{y} + 2y &= 0 \end{aligned}\right\}$$

for the initial conditions $y(0) = x(0) = 0$.

Let us denote by $\bar{x}(p)$, $\bar{y}(p)$ the transforms of the required functions and set up the auxiliary equations:

$$\left.\begin{aligned} 2p\bar{x}(p) + p\bar{y}(p) + \bar{x}(p) &= \frac{1}{p}, \\ p\bar{x}(p) + 3p\bar{y}(p) + 2\bar{y}(p) &= 0. \end{aligned}\right\}$$

Thus, for the transforms we have obtained a linear system of algebraic equations. The determinant of the system

$$\Delta = \begin{vmatrix} 2p+1 & p \\ p & 3p+2 \end{vmatrix} = 5p^2 + 7p + 2.$$

Solving the system of algebraic equations, we find

$$\bar{x}(p) = \frac{3p+2}{p(5p^2+7p+2)}, \quad \bar{y}(p) = \frac{-1}{5p^2+7p+2}.$$

Let us write the transform $\bar{y}(p)$ in the form

$$\bar{y}(p) = -\frac{1}{5}\frac{1}{(p+0.7)^2 - 0.09} = -\frac{2}{3}\frac{0.3}{(p+0.7)^2 - (0.3)^2},$$

whence

$$y(t) = -\frac{2}{3}e^{-0.7t}\sinh(0.3)\,t.$$

Further,

$$\bar{x}(p) = \frac{3}{5[(p+0.7)^2-(0.3)^2]} + \frac{1}{p} - \frac{5p+7}{5p^2+7p+2}$$

$$= 2\,\frac{0.3}{(p+0.7)^2-(0.3)^2} + \frac{1}{p} - \frac{p+0.7}{(p+0.7)^2-(0.3)^2} -$$

$$- \frac{0.7}{(p+0.7)-(0.3)^2},$$

i. e.

$$x(t)$$

$$= 2e^{-0.7t}\sinh(0.3t) + 1 - e^{-0.7t}\cosh(0.3t) - \frac{7}{3}e^{-0.7t}\sinh(0.3t)$$

$$= 1 - \frac{1}{3}e^{-0.7t}\sinh(0.3t) - e^{-0.7t}\cosh(0.3t).$$

Evaluation of Integrals.

Example 5. Evaluate the integral $I(x) = \int\limits_0^\infty \frac{1-\cos xt}{t^2}\,dt$.

Let us find the transform of the integral:

$$L[I(x);\ p]$$

$$= \int\limits_0^\infty e^{-px} \int\limits_0^\infty \frac{1-\cos xt}{t^2}\,dt\,dx = \int\limits_0^\infty \int\limits_0^\infty e^{-px}(1-\cos xt)\,dx\,\frac{dt}{t^2}$$

$$= \int\limits_0^\infty L[1-\cos xt;\ p]\frac{dt}{t^2} = \int\limits_0^\infty \left[\frac{1}{p} - \frac{p}{p^2+t^2}\right]\frac{dt}{t^2}$$

$$= \int\limits_0^\infty \frac{dt}{p(p^2+t^2)} = \frac{1}{p^2}\arctan\frac{t}{p}\Big|_{t=0}^{\infty} = \frac{\pi}{2p^2}.$$

Thus,

$$I(x) = \frac{\pi}{2}x.$$

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This textbook was written as a course in higher mathematics for university students majoring in various fields of natural sciences.

The text consists of twenty six chapters and two appendices. Each chapter contains a large number of test questions and exercises to make the subject matter more readily comprehensible. Answers are provided for all problems in the main text, and every effort has been made to insure that they are accurate.

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